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A BLOOD BANK MODEL WITH PERISHABLE BLOOD AND DEMAND IMPATIENCE[*]

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We consider a stochastic model for a blood bank, in which amounts of blood are offered and demanded according to independent compound Poisson processes. Blood is perishable, i.e., blood can only be kept in storage for a limited amount of time. Furthermore, demand for blood is impatient, i.e., a demand for blood may be canceled if it cannot be satisfied soon enough. For a range of perishability functions and demand impatience functions, we derive the steady-state distributions of the amount of blood X_b kept in storage, and of the amount of demand for blood X_d (at any point in time, at most one of these quantities is positive). Under certain conditions we also obtain the fluid and diffusion limits of the blood inventory process, showing in particular that the diffusion limit process is an Ornstein-Uhlenbeck process.

1. Introduction. This paper is devoted to the study of a stochastic blood bank model in which amounts of blood are offered and demanded according to stochastic processes, and in which blood is perishable (i.e., blood can only be kept for a limited amount of time) and demand for blood is impatient (i.e., a demand for blood may be canceled if it cannot be satisfied soon enough). Let us first provide some background, and subsequently sketch the blood bank model in some more detail.

One of the major issues in securing blood supply to patients worldwide is to provide blood of the best achievable quality, in the needed quantities. In

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most countries blood, which is collected as whole blood units from human donors, is separated into different components which are subsequently stored under different storage conditions according to their biological characteristics, functions and respective expiration dates. Blood units and components are ordered by local hospital blood banks (LBB) from the Central Blood Bank (CBB) respectively) according to their operational needs. The CBB has to run its inventory and supply according to these requests and to the need to keep sufficient stock for immediate release in emergency situations. It also has to perform tests to determine the unit's blood type and to detect the presence of various pathogens which are able to cause transfusion-transmitted diseases, such as Hepatitis B (HBV), Hepatitis C (HCV), Human Immunodeficiency Virus (HIV) and Syphilis; see, e.g., Steiner et al. [27].

Blood consists of several components: Red Blood Cells (RBC), plasma and platelets. In addition there are 8 blood groups (types): $O^+, O^-, A^+, A^-, B^+, B^-, AB^+, AB^-$ ($-$ means Rh negative) where the interrelationship between the transfusion issuing policies among the 8 types is quite intricate. It turns out that each of the negative types can satisfy the corresponding $+$ type, but not vice versa. Blood components are perishable as RBC can be used for only 35 to 42 days and platelets for only 5 days (plasma, however, can be frozen and kept for one year). Accordingly, if RBC and particularly platelets are not used for blood transfusion within their expiration dates then they perish.

In most developed countries demand requirements of about $50,000$ blood donations are needed per one million persons per year. About 95% of these donations are aggregated by CBB's and the remaining 5% by LBB's. Blood units stored at the CBB are usually ordered by LBB's for planned elective surgeries. However, as it happens rather frequently, elective surgeries turn out to become emergency ones due to various conditions of the patient involved. In such cases, hospitals use their own local blood banks to supply the demand, and they cancel the required demand from the CBB; this is what we refer to as demand impatience. A good review on supply chain management of blood products appears in Beliën and Forcé [6] and the references cited therein. Other relevant studies are Ghandforoush and Sen [17] and Stanger et al. [26].

In this paper we consider the analysis of blood perishability and demand impatience, concentrating on only one blood type. We do this by considering the stochastic inventory processes $\{X_b(t), t \geq 0\}$, with $X_b(t)$ the amount of blood kept in storage at time t, and $\{X_d(t), t \geq 0\}$, with $X_d(t)$ the amount of demand for blood (the shortage) at time t. If $X_b(t) > 0$ then $X_d(t) = 0$, and if $X_d(t) > 0$ then $X_b(t) = 0$. We assume that amounts of blood arrive according to a Poisson process, and that requests for blood arrive according to another, independent, Poisson process. The delivered and requested amounts of blood are assumed to be random variables. We represent the perishability of blood by letting the amount of blood, when positive, decrease in a state-dependent way: if the amount is v, then the decrement rate is $\xi_b v + \alpha_b$. The ξ_b factor is motivated by the fact that a large amount of blood suggests that some of the blood has been present for quite a while – and hence there is a relatively high perishability rate when much blood is in inventory. The α_b factor provides additional modeling flexibility. One can in this way represent the blood perishability more accurately; but the α_b term could also, e.g., represent a fluid demand rate of individuals or organizations, which contact the CBB directly, and that is only satisfied when there is inventory. Similarly, we represent the demand impatience by a decrement rate $\xi_d v + \alpha_d$. The ξ_d factor is motivated by the following fact. When there is a large shortage (demand) of blood, there are probably many patients waiting for blood, so many patients that might become impatient (i.e., they could recover, or die, or become in need of emergency surgery) leading to a cancellation of the required demand from the CBB. Again, the α_d factor provides additional modeling flexibility; it not only allows us to represent demand impatience more accurately, but it could also, e.g., represent additional donations of individuals in times of blood shortage. To keep things simple, in the remainder of the paper, we shall refer to the $\xi_b v + \alpha_b$ term as the blood perishability rate, and to the $\xi_d v + \alpha_d$ term as the demand impatience rate.

The inclusion of both the perishability factor $\xi_b v+\alpha_b$ and the demand impatience factor $\xi_d v+\alpha_d$ makes the analysis of the ensuing model mathematically quite challenging, but leads to a very general model that contains many well-known models as special cases. Our two-sided stochastic process, with both jumps upward and jumps downward, and with the rather general slope factors $\xi v+\alpha$, could represent a quite large class of stochastic phenomena. It should for example be noted that this model is a two-sided generalization of the well-known shot-noise model that describes certain physical phenomena (cf. [19]). In some of our calculations we remove either the ξ factors or the α factors, and this results in easier calculations and more explicit results.

Our main results are: (i) Determination of the steady-state distributions of the amounts of blood and of demand in inventory; in particular, we present a detailed analysis of the case in which the delivered and requested amounts of blood are both exponentially distributed. (ii) Expressions for mean amounts of blood and demand in storage, and for the probability of not being able to satisfy demand. (iii) We obtain the fluid and diffusion limits of the blood inventory process, showing in particular that the diffusion limit process is an Ornstein-Uhlenbeck process.

The paper is organized as follows: Section 2 presents a detailed model description. A global analysis of the densities of demand and of blood amount in storage is contained in Section 3. A detailed analysis of the case of exponentially distributed delivered and requested blood amounts, when $\alpha_b = \alpha_d = 0$ (i.e., pure proportionality) is provided in Section 4. In Section 5 we treat the case of Coxian distributed delivered and requested blood amounts, using a Laplace transform approach. Section 6 is devoted to the case $\xi_b = \xi_d = 0$. The fluid and diffusion scalings are discussed in Section 7, and in Section 8 we present numerical results for certain performance measures like mean net amount of blood and the probability that there is a shortage of blood. These results indicate, a.o., that the probability that there is a shortage of blood can be accurately approximated via a Normal approximation, based on the Ornstein-Uhlenbeck process appearance in the diffusion scaling. Section 9 contains some conclusions and suggestions for further research.

2. Model description. We consider the following highly simplified model of a blood bank, restricting ourselves to only one type of blood.

Blood amounts arrive according to a Poisson process with rate λ_b. The amounts which successively arrive are independent, identically distributed random variables $B_1, B_2, \dots$ with distribution $F_b(\cdot)$; $\bar{F}_b(x) = 1 - F_b(x)$.

Demands for blood arrive according to a Poisson process with rate λ_d. The successive demand amounts are independent, identically distributed random variables $D_1, D_2, \dots$ with distribution $F_d(\cdot)$; $\bar{F}_d(x) = 1 - F_d(x)$. We view these amounts as continuous quantities, measured in, e.g., liters.

If there is enough blood to meet demand, then that demand is immediately satisfied. If there is some blood, but not enough to fully satisfy a demand, then that demand is partially satisfied, using all the available blood; the remainder of the demand may be satisfied later.

Blood has a finite expiration date. We make the assumption that if the total amount of blood present is $x > 0$, then blood is discarded – because of its finite expiration date – at a rate $\xi_b x + \alpha_b$, so linear in x.

Blood demands have a finite patience. We make the assumption that if the total amount of demand present is $x > 0$, then demand disappears – because of its finite patience – at a rate $\xi_d x + \alpha_d$, so linear in x. This state-dependent decrement rate results in decay of $X_b(t)$ and $X_d(t)$ that is exponential in time in between arrivals of blood or demand.

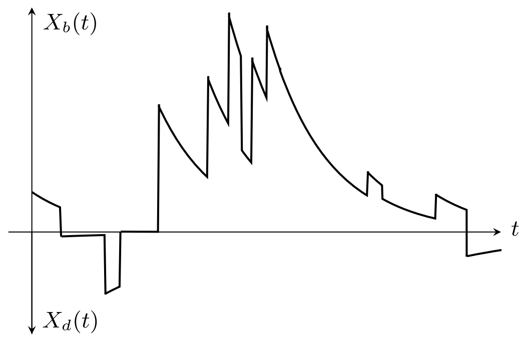


Fig 1. *Sample path of net amount of blood available as a function of time.*

Notice that *either* the total amount of blood present, *or* the total amount of demands, is zero, *or* both are zero; they cannot be both positive. Hence we can easily in one figure depict the two-sided process $\{X(t), t \geq 0\} = \{(X_b(t), X_d(t)), t \geq 0\}$ of total blood and total demand amounts present at any time t, as we have done in Figure 1, which moreover illustrates the exponential decay of demand (cq. blood) over time. For our purposes, we are mainly interested in the characteristics of the process described above in stationarity. Let us denote by X_d the steady-state total amount of demand and by X_b the steady-state total amount of blood present, with corresponding density functions $f(\cdot)$ and $g(\cdot)$, respectively. Notice that these are defective densities; we have $\int_{0+}^{\infty} f(v)\mathrm{d}v = \pi_d = \mathbb{P}(\text{demand} > 0)$ and $\int_{0+}^{\infty} g(v)\mathrm{d}v = \pi_b = \mathbb{P}(\text{blood} > 0)$. If $\alpha_b = \alpha_d = 0$, then neither X_b nor X_d has probability mass at zero, and $\pi_b + \pi_d = 1$ (when there is only a very small amount x present, the "decay" rate $\xi_b x$ or $\xi_d x$ is very small). However, if α_b and/or α_d is positive, then there is a positive probability π_0 of being in 0.

When ξ_d and ξ_b are positive, existence of these steady-state densities is obvious; otherwise, the conditions for the existence of the steady-state distributions require some discussion, cf. Section 6.

3. Analysis of the densities of demand and of blood amount. In this section we present a global approach towards determining $f(\cdot)$ and $g(\cdot)$ in the most general form of our model. Using the Level Crossing Technique (LCT), we derive two integral equations in $f(\cdot)$ and $g(\cdot)$. Before attempting to solve these equations, we consider a few important performance measures which can be expressed in $f(\cdot)$ and $g(\cdot)$, viz., π_0 and the mean length of time during which, uninterruptedly, there is a positive amount of blood (respectively demand). The latter could be viewed as the busy period of the X_b process (respectively of the X_d process).

First, we consider the density $g(\cdot)$ of the amount of blood. We equate the rate at which some positive blood level v is upcrossed and downcrossed, respectively. LCT leads to the following integral equation: for $v > 0$,

$$\begin{aligned}\lambda_b \int_0^v g(y)\bar{F}_b(v-y)\mathrm{d}y + \lambda_b \int_0^\infty f(y)\bar{F}_b(v+y)\mathrm{d}y + \pi_o\lambda_b\bar{F}_b(v) \\ = \lambda_d \int_v^\infty g(y)\bar{F}_d(y-v)\mathrm{d}y + (\xi_b v + \alpha_b)g(v).\end{aligned} \tag{3.1}$$

Here the three terms in the lefthand side represent the rate of crossing level v from below; the first term corresponds to a jump from a blood inventory level between 0 and v, whereas the second term corresponds to a jump from a shortage level, and the third term corresponds to a jump from level 0. The two terms in the righthand side represent the rate of crossing level v from above; the first term corresponds to a jump from above v, and the second term to a smooth crossing.

Next we consider the density $f(\cdot)$ of the amount of demand (shortage). We equate the rate at which some positive demand level v is upcrossed and downcrossed, respectively. LCT leads to the following integral equation: for $v > 0$,

$$\begin{aligned}\lambda_d \int_0^v f(y)\bar{F}_d(v-y)\mathrm{d}y + \lambda_d \int_0^\infty g(y)\bar{F}_d(v+y)\mathrm{d}y + \pi_0\lambda_d\bar{F}_d(v) \\ = \lambda_b \int_v^\infty f(y)\bar{F}_b(y-v)\mathrm{d}y + (\xi_d v + \alpha_d)f(v).\end{aligned} \tag{3.2}$$

It should be noted that these two, coupled, equations are symmetric (swap f and g, and the b and d parameters).

In general, it appears to be very difficult to solve these integral equations. In Section 4 we assume that both $F_b(\cdot)$ and $F_d(\cdot)$ are exponential. In that case we are able to obtain explicit expressions of $f(\cdot)$ and $g(\cdot)$, in terms of hypergeometric functions. In Section 5 we consider the case that $F_b(\cdot)$ and $F_d(\cdot)$ are Coxian distributions, a class of distributions that lies dense in the class of all distributions of non-negative random variables, and that is suitable for handling the above coupled integral equations via Laplace transforms (LT). We are able to transform (3.1) and (3.2) into inhomogeneous first-order differential equations in the LT's of $f(\cdot)$ and $g(\cdot)$, and thus to obtain those LT's.

3.1. *A few simple performance measures.* Without solving (3.1)–(3.2) explicitly, we are able to deduce some characteristics of the steady-state inventory level.

In Proposition 1, we relate π_0 to the densities $f(\cdot)$ and $g(\cdot)$. Subsequently we express the mean length of time during which there is, uninterruptedly, a positive amount of blood present (we call this the non-emptiness period of the inventory system), into $f(\cdot)$, $g(\cdot)$ and π_0. We do the same for the mean length of time during which there is, uninterruptedly, a positive demand, i.e., the non-emptiness period of the demand process, see Proposition 2.

PROPOSITION 1. *Let π_0 be the steady-state atom probability of the zero period. Then*

$$\pi_0 = \frac{\alpha_d f(0) + \alpha_b g(0)}{\lambda_d + \lambda_b}.$$

PROOF. Substitute $\upsilon = 0$ in (3.1) and (3.2) and take the sum. The result is obtained after several steps of elementary algebra. □

The result introduced in the Proposition above is very intuitive. By LCT, $\alpha_d f(0) + \alpha_b g(0)$ is the rate at which level 0 is reached (i.e., the process will now really stay at 0 for a while), so that $[\alpha_d f(0) + \alpha_b g(0)]^{-1}$ is the expected length of time between two successive times level 0 is reached by the fluid. More precisely, the (*zero periods* , *non-zero periods*) generate an alternating renewal process whose expected cycle length is $[\alpha_d f(0) + \alpha_b g(0)]^{-1}$. The expected length of the zero period is $[\lambda_d + \lambda_b]^{-1}$, since the end of the zero period is terminated at the moment of the next jump. But the jump process is a Poisson process with rate $\lambda_d + \lambda_b$. Now the renewal reward theorem simply says that

$$\pi_0 = \frac{\mathbb{E}[\text{zero period}]}{\mathbb{E}[\text{cycle}]}.$$

In preparation of the next proposition, for the process $\mathbf{X} = \{X(t) : t \geq 0\}$ we define a modified process $\mathbf{X_m} = \{X_m(t) : t \geq 0\}$ where $\mathbf{X_m}$ is constructed by deleting the *zero periods* (only the zero periods, not the emptiness periods) from $\mathbf{X}$ and gluing together the *non-zero periods*. The modified process is $\mathbf{X_m}$ such that $X_m(t) = X_d(t)\mathbf{1}_{\{X_d(t)>0\}} + X_b(t)\mathbf{1}_{\{X_b(t)>0\}}$ where by definition of the model $\{X_d(t) > 0\} \Rightarrow \{X_b(t) = 0\}$ and $\{X_b(t) > 0\} \Rightarrow \{X_d(t) = 0\}$.

PROPOSITION 2. *Let B_b and I_b be the generic non-emptiness period and the emptiness period, respectively, of the inventory system. Similarly, let B_d and I_d be the generic non-emptiness period and the emptiness period, respectively, of the demand process. Then*

$$(i) \begin{cases} \mathbb{E}[B_b] = \frac{1-\pi_0}{\alpha_b g(0) + \lambda_d \int_0^\infty \bar{F}_d(y) g(y) \mathrm{d}y}, \\ \mathbb{E}[B_d] = \frac{1-\pi_0}{\alpha_d f(0) + \lambda_b \int_0^\infty \bar{F}_b(y) f(y) \mathrm{d}y}. \end{cases}$$

And

$$(ii)\begin{cases} \mathbb{E}[I_b] = \frac{1}{\lambda_b \int_0^\infty \bar{F}_b(y)f(y)dy + \lambda_b\pi_0} - \mathbb{E}[B_b], \\ \mathbb{E}[I_d] = \frac{1}{\lambda_d \int_0^\infty \bar{F}_d(y)g(y)\mathrm{d}y + \lambda_d\pi_0} - \mathbb{E}[B_d]. \end{cases}$$

PROOF. (i) Consider the non-emptiness period of the inventory system. The steady state densities of the inventory system and the demand process of $\mathbf{X_m}$ are given by

$$g_m(x) = \frac{g(x)}{1-\pi_0}, \qquad f_m(x) = \frac{f(x)}{1-\pi_0},$$

respectively. At the end of the non-emptiness period of the inventory system there are two disjoint ways (disjoint events) to downcross level 0+. Either level 0 is downcrossed by a negative jump or level 0+ is reached by the fluid reduction (both in $\mathbf{X_m}$). The rate of the first event is $\lambda_d \int_0^\infty \bar{F}_d(y)g_m(y)dy$ and by LCT the rate of the second event is $\alpha_b g_m(0)$. Since the events are disjoint, the rate of downcrossings of level 0+ is $\lambda_d \int_0^\infty \bar{F}_d(y)g_m(y)dy + \alpha_b g_m(0)$. That means that the expected length of the non-emptiness period is given by $[\lambda_d \int_0^\infty \bar{F}_d(y)g_m(y)dy + \alpha_b g_m(0)]^{-1}$. Thus

$$\mathbb{E}[B_b] = \frac{1-\pi_0}{\alpha_b g(0) + \lambda_d \int_0^\infty \bar{F}_d(y)g(y)dy}.$$

$\mathbb{E}[B_d]$ is obtained by symmetry.

(ii) Define a *cycle* in the real process $\mathbf{X}$ (not the modified process $\mathbf{X}_m$) as the time between two upcrossings of level 0+. By definition, the emptiness period plus the non-emptiness period is a cycle in $\mathbf{X}$. That means that the expected length of the emptiness period is the expected length of the cycle minus the expected length of the non-emptiness period. The non-emptiness period in $\mathbf{X}$ and in $\mathbf{X}_m$ are identical and the length of the expected cycle is $[\lambda_b \int_0^\infty \bar{F}_b(y)f(y)dy + \lambda_b\pi_0]^{-1}$, since $\lambda_b \int_0^\infty \bar{F}_b(y)f(y)dy + \lambda_b\pi_0$ is the rate of the upcrossings of level 0+. We obtain

$$\mathbb{E}[I_b] + \mathbb{E}[B_b] = \frac{1}{\lambda_b \int_0^\infty \bar{F}_b(y)f(y)dy + \lambda_b\pi_0},$$

yielding $\mathbb{E}[I_b]$. $\mathbb{E}[I_d]$ is obtained by symmetry. □

For the special case in which $\xi_b = \xi_d = \xi$ and $\alpha_b = \alpha_d = 0$, we are able to deduce that the expected steady-state inventory level $\mathbb{E}[X]$ has a simple form.

Proposition 3. *If $\xi_b = \xi_d = \xi$ and $\alpha_b = \alpha_d = 0$, then*

$$\mathbb{E}[X] = m/\xi, \tag{3.3}$$

where $m = \lambda_b \mathbb{E}[B] - \lambda_d \mathbb{E}[D]$.

Proof. We study the discrete-time embedding of the blood inventory process $\{X_k,\ k \geq 1\}$, where X_k denotes the blood inventory level *just before* the k^{th} arrival (either blood or demand). Suppose the process is in steady state. By the PASTA property, we have that $X_k \stackrel{d}{=} X$ for all $k \geq 1$. Also, the process $\{X_k,\ k \geq 1\}$ constitutes a Markov chain, of which the evolution is characterized by the recursion

$$X_{k+1} = (X_k + \mathbb{1}_{k,b} B_k - \mathbb{1}_{k,d} D_k) \cdot \mathbb{E}^{-\xi A_k}, \tag{3.4}$$

where $\mathbb{1}_{k,b}$ and $\mathbb{1}_{k,d}$ denote the indicator function of the event that the k^{th} arrival is a blood or demand arrival, respectively. Remark that the relation holds for both $X_k \geq 0$ and $X_k < 0$. Furthermore, B_k and D_k denote the amount of blood or demand in the k^{th} jump, respectively, and A_k denotes the interarrival time between the k^{th} and $(k+1)^{th}$ arrival. Note that A_k is the minimum of two exponentially distributed random variables with rate λ_b and λ_d, so that A_k is exponentially distributed with rate $\lambda_b + \lambda_d$. Next, we take the expectation on both sides of (3.4), which gives

$$\mathbb{E}[X_{k+1}] = \Big(\mathbb{E}[X_k] + \frac{\lambda_b}{\lambda_b + \lambda_d}\,\mathbb{E}[B] - \frac{\lambda_d}{\lambda_b + \lambda_d}\,\mathbb{E}[D]\Big)\,\mathbb{E}\big[\mathrm{e}^{-\xi A_k}\big]. \tag{3.5}$$

Here, we used independence between Poisson processes and their jump sizes and their memoryless property. Since $X_k \stackrel{d}{=} X$, we have $\mathbb{E}[X_{k+1}] = \mathbb{E}[X_k] = \mathbb{E}[X]$, and thus we may rewrite (3.5) as

$$\mathbb{E}[X] = \left(\mathbb{E}[X] + \frac{\lambda_b \mathbb{E}[B] - \lambda_d \mathbb{E}[D]}{\lambda_b + \lambda_d}\right) \cdot \frac{\lambda_b + \lambda_d}{\lambda_b + \lambda_d + \xi}, \tag{3.6}$$

from which we easily deduce $\mathbb{E}[X] = (\lambda_b \mathbb{E}[B] - \lambda_d \mathbb{E}[D])/\xi = m/\xi$. □

4. The exponential case.

4.1. *Density functions.* We assume in this section that $\bar{F}_b(x) = \mathrm{e}^{-\mu_b x}$ and $\bar{F}_d(x) = \mathrm{e}^{-\mu_d x}$. Let $\rho_d := \lambda_d/\mu_d$ and $\rho_b := \lambda_b/\mu_b$ denote the expected amount of demand requested, and amount of blood delivered into the system, per time unit. Moreover, we take $\alpha_b = \alpha_d = 0$. Under these assumptions, we can solve (3.2) and (3.1) explicitly.

Equations (3.2) and (3.1) reduce to:

$$\lambda_d \int_0^v f(y)\mathrm{e}^{-\mu_d(v-y)}\mathrm{d}y + \lambda_d \mathrm{e}^{-\mu_d v}\int_0^\infty g(y)\mathrm{e}^{-\mu_d y}\mathrm{d}y = \lambda_b \int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y + \xi_d v f(v), \qquad v>0; \tag{4.1}$$

$$\lambda_b \int_0^v g(y)\mathrm{e}^{-\mu_b(v-y)}\mathrm{d}y + \lambda_b \mathrm{e}^{-\mu_b v}\int_0^\infty f(y)\mathrm{e}^{-\mu_b y}\mathrm{d}y = \lambda_d \int_v^\infty g(y)\mathrm{e}^{-\mu_d(y-v)}\mathrm{d}y + \xi_b v g(v), \qquad v>0. \tag{4.2}$$

In our analysis, we concentrate on the derivation of $f(v)$. Notice that, once $f(\cdot)$ has been determined, $g(\cdot)$ follows by swapping parameters (symmetry).

In Appendix A we show how the integral equations (4.1)–(4.2) can be translated into the following decoupled second order differential equations:

$$\xi_d v f''(v) + \left(2\xi_d - \lambda_d - \lambda_b + \mu_d\xi_d v - \mu_b\xi_d v\right) f'(v) + \left(\mu_d\xi_d - \mu_b\xi_d - \mu_d\lambda_b + \mu_b\lambda_d - \mu_b\mu_d\xi_d v\right) f(v) = 0 \tag{4.3}$$

and

$$\xi_b v g''(v) + \left(2\xi_b - \lambda_d - \lambda_b + \mu_b\xi_d v - \mu_d\xi_b v\right) g'(v) + \left(\mu_b\xi_b - \mu_d\xi_b - \mu_b\lambda_d + \mu_d\lambda_b - \mu_d\mu_b\xi_b v\right) g(v) = 0, \tag{4.4}$$

with the additional constraint (obtained by applying the level crossing identity for level $v = 0$ in either (4.1) or (4.2)):

$$\lambda_b \int_0^\infty f(y)\mathrm{e}^{-\mu_b y}\mathrm{d}y = \lambda_d \int_0^\infty g(y)\mathrm{e}^{-\mu_d y}\mathrm{d}y. \tag{4.5}$$

Equation (4.3) describes a known type of second order differential equation, namely the Extended Confluent Hypergeometric Equation [25], which allows an explicit solution. A detailed deduction of the solution to (4.3) is given in Appendix B, and yields the following result.

Proposition 4. *The probability density functions of the amount of demand X_d and the amount of blood present X_b are given by*

$$f(v) = \pi_d \frac{\Gamma\left(1+\frac{\lambda_b}{\xi_d}\right)}{\Gamma\left(\frac{\lambda_b+\lambda_d}{\xi_d}\right)} \frac{e^{-\mu_d v}}{{}_2F_1\left(1-\frac{\lambda_d}{\xi_d}, 1, 1+\frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)} \cdot U\left(1-\frac{\lambda_d}{\xi_d}, 2-\frac{\lambda_b+\lambda_d}{\xi_d}, (\mu_b+\mu_d)v\right), \tag{4.6}$$

$$g(v) = \pi_b \frac{\Gamma\left(1+\frac{\lambda_d}{\xi_b}\right)}{\Gamma\left(\frac{\lambda_b+\lambda_d}{\xi_b}\right)} \frac{e^{-\mu_b v}}{{}_2F_1\left(1-\frac{\lambda_b}{\xi_b}, 1, 1+\frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right)} \cdot U\left(1-\frac{\lambda_b}{\xi_b}, 2-\frac{\lambda_b+\lambda_d}{\xi_b}, (\mu_b+\mu_d)v\right), \tag{4.7}$$

for $v > 0$, respectively.

Here, $\Gamma(\cdot)$ denotes the gamma function, ${}_2F_1(a,b,c,z)$ is the Gaussian hypergeometric function, defined as

$$ {}_2F_1(a,b,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} z^n \tag{4.8}$$

and $U(a,b,z)$ is Tricomi's confluent hypergeometric function, see [25]

$$U(a,b,x) = \frac{\Gamma(b-1)}{\Gamma(1+a-b)} \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} x^n + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} \sum_{n=0}^{\infty} \frac{(1+a-b)_n}{(2-b)_n n!} x^n, \tag{4.9}$$

in which $(a)_n$ is the Pochhammer symbol, defined as $(a)_n = a\cdot(a+1)\cdots(a+n-1)$. As a direct consequence of Proposition 4, we obtain expressions for the LTs $\phi(s) = \int_0^\infty \mathrm{e}^{-sv} f(v)\mathrm{d}v$ and $\gamma(s) = \int_0^\infty \mathrm{e}^{-sv} g(v)\mathrm{d}v$, for $\Re s \geq 0$, through [25, Eq. (3.2.51)], which we state here for future use.

Corollary 1. *The Laplace transforms for X_d and X_b for $\Re s \geq 0$ are given by*

$$\phi(s) = \pi_d \frac{\mu_d}{\mu_d+s} \frac{{}_2F_1\left(1-\frac{\lambda_d}{\xi_d}, 1, 1+\frac{\lambda_b}{\xi_d}, \frac{s-\mu_b}{s+\mu_d}\right)}{{}_2F_1\left(1-\frac{\lambda_d}{\xi_d}, 1, 1+\frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)}, \tag{4.10}$$

$$\gamma(s) = \pi_b \frac{\mu_b}{\mu_b+s} \frac{{}_2F_1\left(1-\frac{\lambda_b}{\xi_b}, 1, 1+\frac{\lambda_d}{\xi_b}, \frac{s-\mu_d}{s+\mu_b}\right)}{{}_2F_1\left(1-\frac{\lambda_b}{\xi_b}, 1, 1+\frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right)}, \tag{4.11}$$

respectively.

Last, we obtain expressions for π_d and π_b.

Corollary 2. *The probability of positive (cq. negative) inventory is given by,*

$$\pi_b = \frac{\rho_b\, {}_2F_1\left(1-\frac{\lambda_b}{\xi_b}, 1, 1+\frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right)}{\rho_d\, {}_2F_1\left(1-\frac{\lambda_d}{\xi_d}, 1, 1+\frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right) + \rho_b\, {}_2F_1\left(1-\frac{\lambda_b}{\xi_b}, 1, 1+\frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right)}, \tag{4.12}$$

(4.13)
$$\pi_d = \frac{\rho_d\, {}_2F_1\left(1-\frac{\lambda_d}{\xi_d}, 1, 1+\frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)}{\rho_d\, {}_2F_1\left(1-\frac{\lambda_d}{\xi_d}, 1, 1+\frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right) + \rho_b\, {}_2F_1\left(1-\frac{\lambda_b}{\xi_b}, 1, 1+\frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right)},$$

respectively.

PROOF. First, we use (4.5), or equivalently, $\lambda_b\phi(\mu_b) = \lambda_d\gamma(\mu_d)$. By filling in $s = \mu_b$ in (4.10),

$$\pi_d\, \frac{\lambda_b\mu_d}{\mu_b+\mu_d}\, {}_2F_1\left(1-\frac{\lambda_d}{\xi_d}, 1, 1+\frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right)^{-1} = \pi_b\, \frac{\lambda_d\mu_b}{\mu_b+\mu_d}\, {}_2F_1\left(1-\frac{\lambda_b}{\xi_b}, 1, 1+\frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right)^{-1}, \tag{4.14}$$

where we used that ${}_2F_1(a, b, c, 0) = 1$. Using the normalization equation, the result readily follows. □

Finally, by substituting these expressions into both (4.6) and (4.10), we obtain the full pdf for the blood inventory process in steady-state.

THEOREM 5. *The steady-state pdf of the net inventory level X is given by*

$$h(v) = \begin{cases} f(-v), & \text{if } v < 0, \\ g(v), & \text{if } v \geq 0, \end{cases} \tag{4.15}$$

where

(4.16)
$$f(v) = \bar{C}^{-1}\, \frac{\Gamma\left(1+\frac{\lambda_b}{\xi_d}\right)}{\Gamma\left(\frac{\lambda_b+\lambda_d}{\xi_d}\right)}\, \rho_d\, e^{-\mu_d v}\, U\left(1-\frac{\lambda_d}{\xi_d}, 2-\frac{\lambda_b+\lambda_d}{\xi_d}, (\mu_b+\mu_d)v\right),$$

(4.17)
$$g(v) = \bar{C}^{-1}\, \frac{\Gamma\left(1+\frac{\lambda_d}{\xi_b}\right)}{\Gamma\left(\frac{\lambda_b+\lambda_d}{\xi_b}\right)}\, \rho_b\, e^{-\mu_b v}\, U\left(1-\frac{\lambda_b}{\xi_b}, 2-\frac{\lambda_b+\lambda_d}{\xi_b}, (\mu_b+\mu_d)v\right),$$

with

(4.18)
$$\bar{C} = \rho_d\, {}_2F_1\left(1-\frac{\lambda_d}{\xi_d}, 1, 1+\frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right) + \rho_b\, {}_2F_1\left(1-\frac{\lambda_b}{\xi_b}, 1, 1+\frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b}\right).$$

Remark 1. *By applying the Pfaff transformation* ${}_2F_1(a,b,c,z) = (1-z)^{-b}\, {}_2F_1\left(c-a,b,c,\frac{z}{1-z}\right)$*, we may reformulate*

$$
{}_2F_1\left(1-\tfrac{\lambda_d}{\xi_d},1,1+\tfrac{\lambda_b}{\xi_d},-\tfrac{\mu_b}{\mu_d}\right) = \tfrac{\mu_d}{\mu_b+\mu_d}\, {}_2F_1\left(\tfrac{\lambda_b+\lambda_d}{\xi_d},1,\tfrac{\lambda_b}{\xi_d},\tfrac{\mu_b}{\mu_b+\mu_d}\right), \tag{4.19}
$$

so that

$$
\pi_d = \frac{\lambda_d\, {}_2F_1\left(\tfrac{\lambda_b+\lambda_d}{\xi_d},1,\tfrac{\lambda_b}{\xi_d},\tfrac{\mu_b}{\mu_b+\mu_d}\right)}{\lambda_d\, {}_2F_1\left(\tfrac{\lambda_b+\lambda_d}{\xi_d},1,\tfrac{\lambda_b}{\xi_d},\tfrac{\mu_b}{\mu_b+\mu_d}\right) + \lambda_b\, {}_2F_1\left(\tfrac{\lambda_b+\lambda_d}{\xi_b},1,\tfrac{\lambda_d}{\xi_b},\tfrac{\mu_d}{\mu_b+\mu_d}\right)}. \tag{4.20}
$$

By also transforming the hypergeometric term in the numerator of (4.6)*, we get an equivalent form of* (4.16)*, namely*

$$
\begin{aligned}
f(v) = \bar{C}_{\text{alt}}^{-1} &\frac{\Gamma\left(1+\frac{\lambda_b}{\xi_d}\right)}{\Gamma\left(\frac{\lambda_b+\lambda_b}{\xi_d}\right)}\, \frac{\lambda_b\mu_b(\mu_b+\mu_d)}{\mu_d}\, e^{-\mu_d v} \\
&\times U\left(1-\tfrac{\lambda_d}{\xi_d}, 2-\tfrac{\lambda_b+\lambda_d}{\xi_d}, (\mu_b+\mu_d)v\right),
\end{aligned} \tag{4.21}
$$

with

$$
\bar{C}_{\text{alt}} = \lambda_d\, {}_2F_1\left(\tfrac{\lambda_b+\lambda_d}{\xi_d},1,\tfrac{\lambda_b}{\xi_d},\tfrac{\mu_b}{\mu_b+\mu_d}\right) + \lambda_b\, {}_2F_1\left(\tfrac{\lambda_b+\lambda_d}{\xi_b},1,\tfrac{\lambda_d}{\xi_b},\tfrac{\mu_d}{\mu_b+\mu_d}\right). \tag{4.22}
$$

As a consequence, (4.10) *is given by*

$$
\phi(s) = \pi_d \frac{{}_2F_1\left(\tfrac{\lambda_b+\lambda_d}{\xi_d},1,\tfrac{\lambda_b}{\xi_d},\tfrac{\mu_b-s}{\mu_b+\mu_d}\right)}{{}_2F_1\left(\tfrac{\lambda_b+\lambda_d}{\xi_d},1,\tfrac{\lambda_b}{\xi_d},\tfrac{\mu_b}{\mu_b+\mu_d}\right)} = \bar{C}_{\text{alt}}^{-1}\, \lambda_d\, {}_2F_1\left(\tfrac{\lambda_b+\lambda_d}{\xi_d},1,\tfrac{\lambda_b}{\xi_d},\tfrac{\mu_b-s}{\mu_b+\mu_d}\right) \tag{4.23}
$$

By close inspection of the derived density functions, we can observe the following on the distribution shape around $z=0$. The confluent hypergeometric function $U(a,b,z)$ has limiting form as $z \to 0$,

$$
U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + \frac{\Gamma(b-1)}{\Gamma(a)}\, z^{1-b} + O(z^{2-b}), \qquad b \le 2, \tag{4.24}
$$

see [20, Subsec. 13.2]. Note that in our model, $b = 2-(\lambda_b+\lambda_d)/\xi_d < 2$ for all parameter settings. Equation (4.24) shows that $U(a,b,z)$ has a singularity at $z=0$ if $\mathrm{Re}(b) > 1$, which in our case translates to $f(v)$ and $g(v)$ being analytic at $v=0$ if $\lambda_b+\lambda_d > \xi_d$ and $\lambda_b+\lambda_d > \xi_b$, respectively. Assuming $\lambda_b+\lambda_d > \max\{\xi_b,\xi_d\}$, (4.24) also implies that

$$
\lim_{v\to 0} f(v) = \bar{C}^{-1}\, \frac{\Gamma\left(1+\frac{\lambda_b}{\xi_d}\right)}{\Gamma\left(\frac{\lambda_b+\lambda_d}{\xi_d}\right)}\, \lambda_d\mu_b \cdot \frac{\Gamma\left(\frac{\lambda_b+\lambda_d}{\xi_d}-1\right)}{\Gamma\left(\frac{\lambda_b}{\xi_d}\right)} \tag{4.25}
$$

$$= \bar{C}^{-1} \frac{\frac{\lambda_b}{\xi_d}}{\frac{\lambda_b+\lambda_d}{\xi_d} - 1} \lambda_d \mu_b = \bar{C}^{-1} \frac{\lambda_b \lambda_d \mu_b \mu_d}{\lambda_b + \lambda_d - \xi_d}.$$

Similarly,

$$\lim_{v \to 0} g(v) = \bar{C}^{-1} \frac{\lambda_b \lambda_d \mu_b \mu_d}{\lambda_b + \lambda_d - \xi_b}. \tag{4.26}$$

By equating these two expressions, we conclude that $\lim_{v\to 0} f(v) = \lim_{v\to 0} g(v) < \infty$, i.e. the overall density function $h(v)$ is continuous at $v = 0$, if and only if $\xi_b = \xi_d$.

The asymptotic behavior of U as $z \to \infty$ is given by [25, p. 60],

$$U(a, b, z) \sim z^{-a}, \quad z \to \infty, \tag{4.27}$$

which implies that the density function tail decays as

$$f(v) \sim C^* e^{-\mu_d v} v^{\lambda_d/\xi_d - 1}, \quad v \to \infty, \tag{4.28}$$

for some constant C^*.

4.2. *Special cases.* Based on the density functions in Theorem 5, we make some comments on their properties, and discuss parameter settings that lead to special cases.

Equation (4.28) suggests that the case $\lambda_d = \xi_d$ is special. Indeed, then (4.10) reduces to

$$\phi(s) = \bar{C}^{-1} \lambda_d \mu_b \frac{\mu_d}{\mu_d + s} = \pi_d \frac{\mu_d}{\mu_d + s}, \tag{4.29}$$

where we used that ${}_2F_1(0, a, b, z) = 1$ for all a, b, z. Hence, conditioned on being positive, the amount of demand present is exponentially distributed with parameter μ_d, regardless of the values of $\lambda_d = \xi_d$, as well as λ_b, ξ_b, and μ_b. If we moreover let $\lambda_b = \xi_b$, then

$$\pi_d = \frac{\lambda_d/\mu_d}{\lambda_b/\mu_b + \lambda_d/\mu_d} = \frac{\rho_d}{\rho_b + \rho_d},$$

and X has exponential distribution both above and below 0, with parameters μ_b and μ_d, respectively.

A second special case arises when the process is symmetric, that is, $\lambda_b = \lambda_d = \lambda$, $\mu_b = \mu_d = \mu$ and $\xi_b = \xi_d = \xi$. Obviously, we get $\pi_b = \pi_d = \frac{1}{2}$ due to the symmetry. If we define $\eta := \lambda/\xi$,

$$f(v) = \frac{\Gamma(1+\eta)\, \mu e^{-\mu v}\, U\left(1-\eta, 2(1-\eta), 2\mu v\right)}{2\,\Gamma(2\eta)\,{}_2F_1\left(2\eta, 1, 1+\eta, \frac{1}{2}\right)} \tag{4.30}$$

$$= \frac{\Gamma(1+\eta)}{2\,\Gamma(2\eta)\,{}_2F_1\left(2\eta,1,1+\eta,\frac{1}{2}\right)}\,\frac{\mu}{2\sqrt{\pi}}\,(2\mu v)^{\eta-\frac{1}{2}}\;K_{\frac{1}{2}-\eta}\,(\mu v)\,,$$

where $K_\alpha(\cdot)$ is the modified Bessel function of the second kind, see [20, Eq. (13.6.10)].

4.3. *Performance measures.* Based on Theorem 5, we can directly derive a couple of characteristics of the process. First, we consider the mean inventory level

Corollary 3. *The expected amount of demand (blood) present, given that it is positive equals*

(4.31)
$$\mathbb{E}[X_d|X_d>0] = \frac{1}{\xi_d}\left[\rho_d-\rho_b+\rho_b\,{}_2F_1\left(1-\frac{\lambda_d}{\xi_d},1,1+\frac{\lambda_b}{\xi_d},-\frac{\mu_b}{\mu_d}\right)^{-1}\right],$$

(4.32)
$$\mathbb{E}[X_b|X_b>0] = \frac{1}{\xi_b}\left[\rho_b-\rho_d+\rho_d\,{}_2F_1\left(1-\frac{\lambda_b}{\xi_b},1,1+\frac{\lambda_d}{\xi_b},-\frac{\mu_d}{\mu_b}\right)^{-1}\right].$$

Accordingly, the expected net amount of blood present equals

$$\mathbb{E}[X] = (\rho_b-\rho_d)\left(\frac{\pi_b}{\xi_b}+\frac{\pi_d}{\xi_d}\right)+\frac{\lambda_b\lambda_d}{\bar{C}}\left(\frac{1}{\xi_b}-\frac{1}{\xi_d}\right). \tag{4.33}$$

Proof. We derive the expression in (4.31) through differentiation of (4.10). Let us use shorthand notation

$$F(s) = \left(1-\frac{\lambda_d}{\xi_d},1,1+\frac{\lambda_b}{\xi_d},\frac{s-\mu_b}{s+\mu_d}\right),$$

so that

$$\phi(s) = \pi_d\,\frac{\mu_b}{\mu_b+s}\,\frac{F(s)}{F(0)}.$$

Through [20, (15.5.20)],

$$\frac{\mathrm{d}}{\mathrm{d}z}\,{}_2F_1(a,1,c,z) = \frac{c-1}{z(1-z)}+\frac{1-c+az}{z(1-z)}\,{}_2F_1(a,1,c,z), \tag{4.34}$$

where we also used that ${}_2F_1(a,1,c,z)=1$. Then,

$$\frac{\phi'(0)}{\pi_d} = \left[\frac{-\mu_d}{(\mu_d+s)^2}\,\frac{F(s)}{F(0)}+\frac{\mu_d}{\mu_d+s}\,\frac{F'(s)}{F(0)}\right]_{s=0} = -\frac{1}{\mu_d}+\frac{F'(0)}{F(0)}.$$

By (4.34), we find

$$
\begin{aligned}
F'(s) &= \Big(\frac{\lambda_b/\xi_d}{\frac{s-\mu_b}{s+\mu_d} \cdot \frac{\mu_b+\mu_d}{s+\mu_d}} + \frac{-\lambda_b/\xi_d + (1-\lambda_d/\xi_d)\frac{s-\mu_b}{s+\mu_d}}{\frac{s-\mu_b}{s+\mu_d} \cdot \frac{\mu_b+\mu_d}{s+\mu_d}} F(s) \Big) \frac{\mathrm{d}}{\mathrm{d}s}\Big[\frac{s-\mu_b}{s+\mu_d}\Big] \\
&= \Big(\frac{\lambda_b}{\xi_d} + \Big[\frac{-\lambda_b}{\xi_d} + \Big(1 - \frac{\lambda_d}{\xi_d}\Big) \frac{s-\mu_b}{s+\mu_d} \Big] F(s) \Big) \frac{(s+\mu_d)^2}{(s-\mu_b)(\mu_b+\mu_d)} \cdot \frac{\mu_b+\mu_d}{(s+\mu_d)^2} \\
&= \Big(\frac{\lambda_b}{\xi_d} + \Big[-\frac{\lambda_b}{\xi_d} + \Big(1 - \frac{\lambda_d}{\xi_d}\Big) \frac{s-\mu_b}{s+\mu_d} \Big] F(s) \Big) \frac{1}{s-\mu_b},
\end{aligned}
$$

so that

$$
\begin{aligned}
F'(0) &= -\frac{\lambda_d/\mu_b}{\xi_d} + \left(\frac{\lambda_d/\mu_b}{\xi_d} + \frac{1}{\mu_d} - \frac{\lambda_d/\mu_d}{\xi_d} \right) F(0) \\
&= -\frac{\rho_b}{\xi_d} + \left(\frac{\rho_b - \rho_d}{\xi_d} + \frac{1}{\mu_d} \right) F(0).
\end{aligned}
$$

Hence, we find

$$
\begin{aligned}
\mathbb{E}[X_d | X_d > 0] &= -\frac{\phi'(0)}{\pi_d} = \frac{1}{\mu_d} - \frac{1}{F(0)} \left[-\frac{\rho_b}{\xi_d} + \left(\frac{\rho_b - \rho_d}{\xi_b} + \frac{1}{\mu_d} \right) F(0) \right] \\
&= \frac{1}{\xi_d} \left(\rho_d - \rho_b + \rho_b / F(0) \right) = \frac{1}{\xi_d} \left(-m + \rho_b / F(0) \right),
\end{aligned}
$$

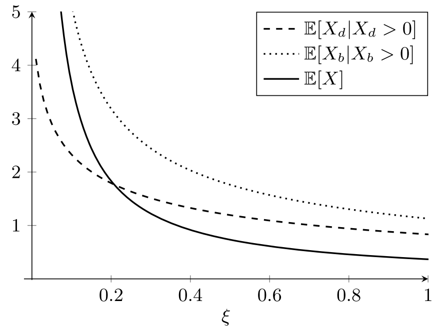
which equals (4.31). The expression for (4.32) follows by symmetry. Furthermore,

$$
\begin{aligned}
\mathbb{E}[X] &= \pi_b \mathbb{E}[X_b | X_b > 0] + \pi_d \mathbb{E}[-X_d | X_d > 0] \\
&= m \left[\frac{\pi_b}{\xi_b} + \frac{\pi_d}{\xi_d} \right] + \frac{\lambda_d}{\mu_d \, \xi_b} \frac{\pi_b}{{}_2F_1 \left(1 - \frac{\lambda_b}{\xi_b}, 1, 1 + \frac{\lambda_d}{\xi_b}, -\frac{\mu_d}{\mu_b} \right)} \\
&\quad - \frac{\lambda_b}{\mu_b \, \xi_d} \frac{\pi_d}{{}_2F_1 \left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d} \right)}.
\end{aligned}
$$

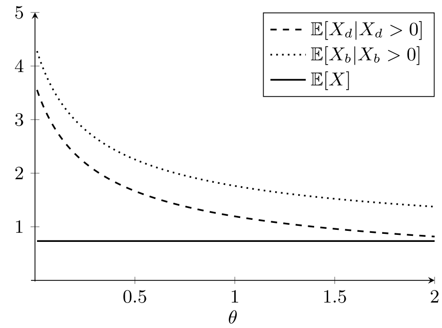
Note that $\pi_d \, {}_2F_1 \left(1 - \frac{\lambda_d}{\xi_d}, 1, 1 + \frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d} \right)^{-1} = \lambda_d \mu_b \bar{C}^{-1}$. Hence,

$$
\mathbb{E}[X] = m \left[\frac{\pi_b}{\xi_b} + \frac{\pi_d}{\xi_d} \right] + \frac{\lambda_b \lambda_d}{\bar{C}} \left(\frac{1}{\xi_b} - \frac{1}{\xi_d} \right),
$$

which completes the proof. Note that we could alternatively have obtained (4.31) through Equation (5.8) below with $s = 0$, since $\mathbb{E}[X_d | X_d > 0] = -\phi'(0)/\phi(0)$. □



(a) As a function of ξ



(b) As a function of θ

FIG 2. *Expected mean amount of blood, demand, and net blood present.*

REMARK 2. *Note that if $\xi_b = \xi_d = \xi$, we get $\mathbb{E}[X] = m(\pi_b + \pi_d)/\xi = m/\xi$, which is consistent with Proposition 3. The expression in (4.31) contains no ξ_b. Indeed, while the value of ξ_b influences the probability that $X_d > 0$, it does not influence the mean of X_d given that $X_d > 0$.*

In Figure 2, we plot the behavior of the three performance metrics in Corollary 3 while keeping m fixed. In Figure 2(a) we set $\lambda_b = 1.2$, $\lambda_d = 1$, $\mu_b = 1$, $\mu_d = 1.2$, so that $m = 11/30$ and vary $\xi_b = \xi_d = \xi$ between 0 and 1. In Figure 2b, we fix $\xi_b = \xi_d = 0.5$ and take $\lambda_b = 1.2\theta$, $\lambda_d = \theta$, $\mu_b = \theta$, $\mu_d = 1.2\theta$, so that still $m = 11/30$, and vary θ. Observe that in Figure 2b, $\mathbb{E}[X]$ is constant, since the value of m/ξ if unaffected by the parameter θ.

The last relevant performance indicator we consider is the fraction of demand that is immediately satisfied from stock.

COROLLARY 4. *The probability that a demand request can be fully satisfied from stock is given by*

(4.35)
$$\mathbb{P}(\text{demand satisfied}) = \bar{C}^{-1}\lambda_b\mu_d\left({}_2F_1\left(1-\frac{\lambda_d}{\xi_d}, 1, 1+\frac{\lambda_b}{\xi_d}, -\frac{\mu_b}{\mu_d}\right) - \frac{\mu_b}{\mu_b+\mu_d}\right).$$

PROOF. Using the PASTA property of the Poisson process, we get

$$\begin{aligned}\mathbb{P}(\text{demand satisfied}) &= \mathbb{P}(X > D) = \mathbb{P}(X_b > D)\\ &= \int_0^\infty g(u)(1-\mathrm{e}^{-\mu_d u})\,\mathrm{d}u = \pi_b - \gamma(\mu_d).\end{aligned}$$

Substituting the expressions for π_b as in Corollary 2 and $\gamma(\mu_b)$ as in (4.11) yields the result. ☐

5. The general case – A Laplace transform approach. In this section we outline how the integral equations (3.1) and (3.2) can be solved using Laplace transforms, when we make the restriction that $F_b(\cdot)$ and $F_d(\cdot)$ are Coxian distributions. This is not a major restriction, because the class of Coxian distributions lies dense in the class of all distributions of nonnegative random variables (cf. [3], Section III.4); hence one can approximate $F_b(\cdot)$ arbitrarily closely by a Coxian distribution.

If X_i, $i = 1, 2, \ldots, K$ are independent, exponentially distributed random variables, and $\mathbb{E}[X_i] = \frac{1}{\beta_i}$, $i = 1, 2, \ldots, K$, then a Coxian amount of blood B can be represented as:

$$B = \sum_{j=1}^{i} X_j \text{ with probability } p_i \prod_{j=1}^{i-1}(1 - p_j), \quad i = 1, 2, \ldots, K. \tag{5.1}$$

In the above case, it is easily verified that one can represent $\bar{F}_b(x)$ as follows:

$$\bar{F}_b(x) = \mathbb{P}(B > x) = \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1}(1 - p_h) \sum_{j=1}^{i} \mathrm{e}^{-\beta_j x} \prod_{l=1;l\neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j}, \tag{5.2}$$

if all β_j are different. If two β_j coincide, then a term with $x\mathrm{e}^{-\beta_j x}$ (Erlang-2) must be added. We leave this to the reader, but in Remark 2 of [4] we outline how Erlang terms can be handled in solving the integral equations (3.2) and (3.1). The counterpart of (5.2) for the case that $F_d(\cdot)$ is Coxian, is

$$\bar{F}_d(x) = \mathbb{P}(D > x) = \sum_{i=1}^{L} q_i \prod_{h=1}^{i-1}(1 - q_h) \sum_{j=1}^{i} \mathrm{e}^{-\delta_j x} \prod_{l=1;l\neq j}^{i} \frac{\delta_l}{\delta_l - \delta_j}. \tag{5.3}$$

Taking Laplace transforms $\phi(s) = \int_0^\infty \mathrm{e}^{-sy} f(y)\mathrm{d}y$ and $\gamma(s) = \int_0^\infty \mathrm{e}^{-sy} g(y)\mathrm{d}y$ in (3.1) and (3.2) results in first-order inhomogeneous differential equations in $\phi(s)$ and $\gamma(s)$, respectively, which can be solved in a straightforward way.

$$\phi'(s) = A_H(s)\phi(s) + A_I(s), \tag{5.4}$$

with the homogeneous term $A_H(s)$ being given by

$$\begin{aligned} A_H(s) := -\frac{1}{\xi_d}\Bigg[&\lambda_d \sum_{i=1}^{L} q_i \prod_{h=1}^{i-1}(1 - q_h) \sum_{j=1}^{i} \frac{1}{\delta_j + s} \prod_{l=1;l\neq j}^{i} \frac{\delta_l}{\delta_l - \delta_j} \\ &- \lambda_b \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1}(1 - p_h) \sum_{j=1}^{i} \frac{1}{\beta_j - s} \prod_{l=1;l\neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} - \alpha_d\Bigg], \end{aligned} \tag{5.5}$$

and the inhomogeneous term $A_I(s)$ being given by

(5.6)
$$A_I(s) := -\frac{1}{\xi_d}\left[\lambda_d \sum_{i=1}^{L} q_i \prod_{h=1}^{i-1}(1-q_h)\sum_{j=1}^{i}\frac{1}{\delta_j+s}[\gamma(\delta_j)+\pi_0]\prod_{l=1;l\neq j}^{i}\frac{\delta_l}{\delta_l-\delta_j}\right.$$
$$\left. + \lambda_b \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1}(1-p_h)\sum_{j=1}^{i}\frac{1}{\beta_j-s}\phi(\beta_j)\prod_{l=1;l\neq j}^{i}\frac{\beta_l}{\beta_l-\beta_j}\right].$$

The solution of (5.4) is given by the following expression:

$$\phi(s) = \phi(0)\mathrm{e}^{\int_0^s A_H(z)\mathrm{d}z} + \int_0^s A_I(u)\mathrm{e}^{\int_u^s A_H(z)\mathrm{d}z}\mathrm{d}u, \qquad s \geq 0. \tag{5.7}$$

$\gamma(s)$ is given by a mirror expression, where $\phi(0)$ is replaced by $\gamma(0)$ and where $A_H(s)$ and $A_I(s)$ are replaced by expressions in which K and L are swapped, and p and q, and β_i and δ_i.

In this Coxian case, $A_H(s)$ and $A_I(s)$ are rational functions of s, and the exponent in the integral in (5.7) takes the shape of a sum of terms $(\frac{\beta_j-u}{\beta_j-s})^{\zeta_j} \times (\frac{\delta_j+s}{\delta_j+u})^{\eta_j}$, for some constants ζ_j, η_j. The last term in (5.7) thus contains an integral with various cross products of terms like $\frac{1}{\delta_i+u}(\beta_j-u)^{\zeta_j}(\frac{1}{\delta_j+u})^{\eta_j}$. In (5.10) below we demonstrate this for the special case of exponentially distributed amounts of blood and demand.

It should be noticed that $\phi(0)$, $\gamma(0)$ and π_0 still have to be determined. Furthermore, it should be noticed that $A_H(s)$ and $A_I(s)$ have singularities at $s = \beta_1, \ldots, \beta_K$. These singularities are removable, but handling Formula (5.7) clearly requires some care. Instead of working out the details, we shall below return to the case of exponentially distributed amounts of blood and demand – so $K = L = 1$. For that case we shall not only work out the solution of the differential equation for $\phi(s)$ in detail, including the determination of the missing constants, but we also relate the results to those obtained in Section 4 without resorting to Laplace transforms. Taking $K = 1, p_1 = 1, \delta_1 = \mu_d$, and $L = 1, q_1 = 1, \beta_1 = \mu_b$, we obtain the following two inhomogeneous first order differential equations in the LTs $\phi(s)$ and $\gamma(s)$:

$$\phi'(s) = \phi(s)\left[\frac{\lambda_b}{\xi_d}\frac{1}{\mu_b-s} - \frac{\lambda_d}{\xi_d}\frac{1}{\mu_d+s}\right] - \frac{\lambda_b}{\xi_d}\frac{\phi(\mu_b)}{\mu_b-s} - \frac{\lambda_d}{\xi_d}\frac{\gamma(\mu_d)}{\mu_d+s}, \tag{5.8}$$

$$\gamma'(s) = \gamma(s)\left[\frac{\lambda_d}{\xi_b}\frac{1}{\mu_d-s} - \frac{\lambda_b}{\xi_b}\frac{1}{\mu_b+s}\right] - \frac{\lambda_d}{\xi_b}\frac{\gamma(\mu_d)}{\mu_d-s} - \frac{\lambda_b}{\xi_b}\frac{\phi(\mu_b)}{\mu_b+s}. \tag{5.9}$$

They are routinely solved:

$$\begin{aligned} \phi(s) = \left(\frac{\mu_b}{\mu_b - s}\right)^{\frac{\lambda_b}{\xi_d}} \left(\frac{\mu_d}{\mu_d + s}\right)^{\frac{\lambda_d}{\xi_d}} \Big[\phi(0) \\ - \frac{\lambda_d}{\xi_d}\gamma(\mu_d) \int_0^s \left(\frac{\mu_b - z}{\mu_b}\right)^{\frac{\lambda_b}{\xi_d}} \left(\frac{\mu_d + z}{\mu_d}\right)^{\frac{\lambda_d}{\xi_d}-1} \frac{\mathrm{d}z}{\mu_d} \\ - \frac{\lambda_b}{\xi_d}\phi(\mu_b) \int_0^s \left(\frac{\mu_b - z}{\mu_b}\right)^{\frac{\lambda_b}{\xi_d}-1} \left(\frac{\mu_d + z}{\mu_d}\right)^{\frac{\lambda_d}{\xi_d}} \frac{\mathrm{d}z}{\mu_b} \ \Big]. \end{aligned} \tag{5.10}$$

Similarly,

$$\begin{aligned} \gamma(s) = \left(\frac{\mu_d}{\mu_d - s}\right)^{\frac{\lambda_d}{\xi_b}} \left(\frac{\mu_b}{\mu_b + s}\right)^{\frac{\lambda_b}{\xi_b}} \Big[\gamma(0) \\ - \frac{\lambda_b}{\xi_b}\phi(\mu_b) \int_0^s \left(\frac{\mu_d - z}{\mu_d}\right)^{\frac{\lambda_d}{\xi_b}} \left(\frac{\mu_b + z}{\mu_b}\right)^{\frac{\lambda_b}{\xi_b}-1} \frac{\mathrm{d}z}{\mu_b} \\ - \frac{\lambda_d}{\xi_b}\gamma(\mu_d) \int_0^s \left(\frac{\mu_d - z}{\mu_d}\right)^{\frac{\lambda_d}{\xi_b}-1} \left(\frac{\mu_b + z}{\mu_b}\right)^{\frac{\lambda_b}{\xi_b}} \frac{\mathrm{d}z}{\mu_d} \ \Big]. \end{aligned} \tag{5.11}$$

Notice that the exponents in the above integrals have powers which are larger than -1 (e.g., $\frac{\lambda_d}{\xi_d} - 1$), so that these integrals do not lead to singularities. We still need to determine the two constants $\phi(0) = \pi_d$ and $\gamma(0) = \pi_b$. Together with $\phi(\mu_b)$ and $\gamma(\mu_d)$, we have four unknowns. We determine these unknowns using the following four equations: (i) From (4.5), we get $\lambda_b\phi(\mu_b) = \lambda_d\gamma(\mu_d)$, while (ii) $\pi_d + \pi_b = 1$. Finally, we take (iii) $s = \mu_b$ in (5.10) and (iv) $s = \mu_d$ in (5.11).

Notice that the identity $\lambda_b\phi(\mu_b) = \lambda_d\gamma(\mu_d)$ allows us to reduce the two integrals in (5.10) to one integral (and similarly in (5.11)):

$$\begin{aligned} \phi(s) = \left(\frac{\mu_b}{\mu_b - s}\right)^{\frac{\lambda_b}{\xi_d}} \left(\frac{\mu_d}{\mu_d + s}\right)^{\frac{\lambda_d}{\xi_d}} \Big[\phi(0) \\ - \frac{\lambda_d}{\xi_d}\gamma(\mu_d) \frac{\mu_b + \mu_d}{\mu_b\mu_d} \int_0^s \left(\frac{\mu_b - z}{\mu_b}\right)^{\frac{\lambda_b}{\xi_d}-1} \left(\frac{\mu_d + z}{\mu_d}\right)^{\frac{\lambda_d}{\xi_d}-1} \mathrm{d}z \ \Big]. \end{aligned} \tag{5.12}$$

REMARK 3. *We have numerically verified that the expressions in (5.10) and (4.10) coincide.*

REMARK 4. *If $\lambda_b = 0$ then we have a known queueing model or shot-noise model with state-dependent service rate; cf. Keilson & Mermin [19] and Bekker et al. [5] for the so-called shot noise model.*

Remark 5. *The case $\lambda_d = \xi_d$ is special. Formula (5.10) now reduces to*

$$\phi(s) = \left(\frac{\mu_b}{\mu_b - s}\right)^{\frac{\lambda_b}{\lambda_d}} \frac{\mu_d}{\mu_d + s}\Big[\phi(0) \ - \gamma(\mu_d)\int_0^s \left(\frac{\mu_b - z}{\mu_b}\right)^{\frac{\lambda_b}{\lambda_d}} \frac{\mathrm{d}z}{\mu_d}$$
$$- \frac{\lambda_b}{\lambda_d}\phi(\mu_b)\int_0^s \left(\frac{\mu_b - z}{\mu_b}\right)^{\frac{\lambda_b}{\lambda_d}-1} \frac{\mu_d + z}{\mu_d}\frac{\mathrm{d}z}{\mu_b} \ \Big].$$

Both integrals are easily evaluated (rewrite, in the last integral, $\mu_d + z = \mu_d + \mu_b - (\mu_b - z)$). We find:

$$\phi(s) = \left(\frac{\mu_b}{\mu_b - s}\right)^{\frac{\lambda_b}{\lambda_d}} \frac{\mu_d}{\mu_d + s} \cdot \left[\phi(0) + \frac{\gamma(\mu_d)}{\mu_d}\frac{\lambda_d}{\lambda_b + \lambda_d}\mu_b - \phi(\mu_b)\frac{\mu_d + \mu_b}{\mu_d} - \frac{\phi(\mu_b)}{\mu_d}\frac{\lambda_b}{\lambda_b + \lambda_d}\mu_b\right]$$
$$+ \frac{\mu_d}{\mu_d + s}\Big[\frac{\gamma(\mu_d)}{\mu_d}\frac{\lambda_d(\mu_b - s)}{\lambda_b + \lambda_d} + \phi(\mu_b)\frac{\mu_d + \mu_b}{\mu_d} - \frac{\phi(\mu_b)}{\mu_d}\frac{\lambda_b}{\lambda_b + \lambda_d}(\mu_b - s)\Big]. \tag{5.13}$$

Now observe, cf. (4.5), that $\lambda_b\phi(\mu_b) = \lambda_d\gamma(\mu_d)$. Hence, in both lines of the above formula, two terms cancel. Moreover, $\phi(s)$ should be analytic for $s = \mu_b$, yielding

$$\phi(0) = \phi(\mu_b)\frac{\mu_d + \mu_b}{\mu_d}. \tag{5.14}$$

Finally we obtain (cf. also (4.29)):

$$\phi(s) = \frac{\mu_d}{\mu_d + s}\phi(\mu_b)\frac{\mu_d + \mu_b}{\mu_d} = \phi(0)\frac{\mu_d}{\mu_d + s} = \pi_d\frac{\mu_d}{\mu_d + s}, \tag{5.15}$$

and hence

$$f(x) = \pi_d\mu_d \mathrm{e}^{-\mu_d x}, \quad x > 0; \tag{5.16}$$

the shortage (amount of demand present) is exponentially distributed when $\lambda_d = \xi_d$.

It should be noticed that, if $\lambda_d = \xi_d$, then the first and last term of (4.1) are equal when (5.16) holds; and using (4.5) it is also readily verified that the second and third term of (4.1) are equal. The constant π_d will in general still depend on the parameters $\lambda_d = \xi_d$, λ_b, μ_b and ξ_b.

We end this remark with the observation that in the one-sided shot-noise process (so $\lambda_b = 0$), Bekker et al. [5] also observe that $\lambda_d = \xi_d$ results in an exponential density.

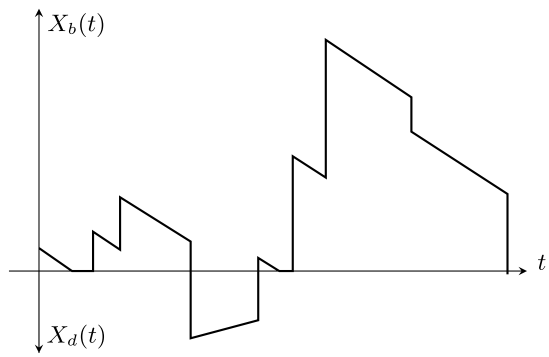


FIG 3. *Sample path of the net amount of blood present if $\xi_b = \xi_d = 0$.*

6. A variant. In this section we assume that the expiration rate of blood and the patience rate of demand are constant; so we take $\xi_b = \xi_d = 0$. A visualization of a possible sample path is depicted in Figure 3.

We again restrict ourselves to the case of exponentially distributed amounts of demand and of blood deliveries. We now need to impose stability conditions. In the case of positive demand, the drift is towards zero if $\lambda_d \mathbb{E}D < \alpha_d + \lambda_b \mathbb{E}B$, while in the case of a positive amount of blood, the drift is towards zero if $\lambda_b \mathbb{E}B < \alpha_b + \lambda_d \mathbb{E}D$. If these two conditions are violated, either the amount of demand or the amount of blood present increases without bound (see also below). In this case, (4.3) reduces to

(6.1)
$$\alpha_d f''(v) + (-\lambda_d - \lambda_b + \mu_d\alpha_d - \mu_b\alpha_d) f'(v) + (-\mu_d\lambda_b + \mu_b\lambda_d - \mu_b\mu_d\alpha_d) f(v) = 0.$$

Hence $f(\cdot)$ is a mixture of two exponential terms: $f(v) = R_+ \mathrm{e}^{-x_+ v} + R_- \mathrm{e}^{-x_- v}$, where x_+ and x_- are the positive and negative root of the equation

$$\alpha_d x^2 - (\mu_d\alpha_d - \mu_b\alpha_d - \lambda_d - \lambda_b)x + (-\mu_d\lambda_b + \mu_b\lambda_d - \mu_b\mu_d\alpha_d) = 0. \quad (6.2)$$

Notice that the last term in the lefthand side of (6.2) is negative if the stability condition $\lambda_d \mathbb{E}D < \alpha_d + \lambda_b \mathbb{E}B$ holds, i.e., if $\mu_b\lambda_d < \mu_d\lambda_b + \mu_b\mu_d\alpha_d$, thus guaranteeing that the product of the two roots x_+ and x_- is negative, and hence that there is a positive and a negative root. One should subsequently observe that R_- must be zero to have a probability density. Hence $f(v)$ is simply (a constant times) an exponential; similarly for $g(v)$. In addition, the steady-state amounts of demand and of blood have an atom at 0 (since ξ_d and ξ_b are no longer zero, the demand and blood processes can reach 0). Interestingly, the model of this section is closely related to the model with workload removal that is considered in [8]. There an $M/G/1$ queue is studied with the extra feature that, at Poisson epochs, a stochastic amount of work is removed. In the $M/M/1$ case with removal of exponential amounts of work (cf. Section 5.1 of [8]), one has the model of the present section when we concentrate on the amount of demand present. One difference with the model in [8] is that, when the workload in that model has become zero, the work becomes positive at rate λ_d, whereas in the present model the amount of blood can become positive (so zero demand is present) and the amount of demand does not have to become positive when demands arrive (because they are immediately satisfied; cf. Fig. 3). So the atom at zero is in the present model larger than in the model of [8]. In our model a positive demand level may be reached from below zero (by a jump, i.e., a demand arriving at an epoch that there is some, but not enough, blood present). The memoryless property of the exponential demand requirement distribution implies that this jump results in a demand level that is $\exp(\mu_d)$, just as if the initial demand level had been zero. In the case of non-exponential demand requirements, our model becomes equivalent with an $M/G/1$ queue with exponential amounts of work removed, and with the special feature that the first service requirement of a busy period has a different distribution. Lemmas 4.1 and 4.2 of [8] present the stability condition of that $M/G/1$ queue with work removal; it amounts to $\lambda_d \mathbb{E}D < \alpha_d + \lambda_b \mathbb{E}B$, which indeed is one of the two stability conditions of the present demand/blood model.

Finally we observe that Formula (5.1) of [8] coincides with (6.2) (take $\alpha_d = 1$, $\lambda_d = \lambda_+$, $\lambda_b = \lambda_-$, $\mu_d = 1/\beta$ and $\mu_b = 1/\gamma$).

7. Asymptotic analysis.

We finally study the model with $\alpha_b = \alpha_d = 0$ from an asymptotic perspective, by obtaining the fluid and diffusion limits of the blood inventory process. That is, we will create a sequence of processes, indexed by $n = 1, 2, ...$, in which we let the rates of blood and demand arrivals grow large. If we then scale the process in a proper manner, we are able to deduce a non-degenerate limiting process, that provides insight in the overall behavior of the arrival volume when the system grows large, which only relies on the first two moments of the blood and demand distributions.

7.1. *Identification of the limiting process.* First, we introduce some additional notation. Let $X_b(t)$ and $X_d(t)$ denote the amount of blood and demand, respectively, at time $t > 0$. Let

$$X(t) := X_b(t) - X_d(t), \tag{7.1}$$

be the net amount of blood available at time t. Remember that $X_b(t), X_d(t) \geq 0$, and $X_b(t) > 0$ *or* $X_d(t) > 0$ for all t, since $\alpha_d = \alpha_b = 0$. Let $N_b(t)$, $N_d(t)$ be the two independent Poisson processes counting the number of arrivals of blood and demand, respectively. Then the following integral representation holds for $X(t)$,

$$X(t) = X(0) - \xi_b \int_0^t X_b(s)\,\mathrm{d}s + \xi_d \int_0^t X_d(s)\,\mathrm{d}s + \sum_{i=1}^{N_b(t)} B_i - \sum_{i=1}^{N_d(t)} D_i. \tag{7.2}$$

For the sake of exhibition, we will concentrate on the case $\xi_b = \xi_d =: \xi$. Our analysis may be extended to the general case. A sketch of this generalization is given at the end of this section without going into the technical difficulties that arise when rigorously proving these limits.

Define

$$Z(t) = \sum_{i=1}^{N_b(t)} B_i - \sum_{i=1}^{N_d(t)} D_i, \tag{7.3}$$

that is, the difference between two compound Poisson processes, so that (7.2) reduces to

$$X(t) = X(0) - \xi \int_0^t X(s)\,\mathrm{d}s + Z(t). \tag{7.4}$$

The first step in the definition of the sequence of processes under investigation is defining the asymptotic scheme we are interested in. As mentioned above, we intend to let the arrival rates grow to infinity. Therefore, in the n^{th} process $X_n(t)$, we replace the rates of the arrival processes by $n\lambda_b$ and $n\lambda_d$. This induces Poisson processes $N_b^{(n)}$ and $N_d^{(n)}$ with arrival rates $n\lambda_b$ and $n\lambda_d$, respectively. However, we have for its marginals

$$N_b^{(n)}(t) \stackrel{d}{=} N_b(nt) \qquad \text{and} \qquad N_d^{(n)}(t) \stackrel{d}{=} N_d(nt), \tag{7.5}$$

so that the term $Z(t)$ in (7.4) in this asymptotic scheme can be replaced by

$$Z_n(t) = \sum_{i=1}^{N_b(nt)} B_i - \sum_{i=1}^{N_d(nt)} D_i. \tag{7.6}$$

The first step in our analysis is obtaining the fluid limit of the process. Bearing in mind application of the Functional Law of Large Numbers (FLLN), we scale the process as $\bar{X}_n(t) = X_n(t)/n$, so that with (7.4)

$$\bar{X}_n(t) = \bar{X}_n(0) - \xi \int_0^t \bar{X}_n(s)\,\mathrm{d}s + \bar{Z}_n(t), \tag{7.7}$$

where $\bar{Z}_n(t) = Z_n(t)/n$.

The essential step in establishing a result on the convergence of $\bar{X}_n$ is the application of [21, Thm. 4.1], which we cite here for completeness, slightly rewritten to fit our setting.

THEOREM 6 ([21, Thm. 4.1]). *Let $D[0,\infty)$ be the space of all one-dimensional real-valued càdlàg functions defined on $[0,\infty)$, endowed with the usual J_1-Skorohod topology. Consider the integral representation*

$$x(t) = y(t) + \int_0^t u(x(s))\,\mathrm{d}s, \qquad t \geq 0, \tag{7.8}$$

where $u : \mathbb{R} \to \mathbb{R}$ satisfies $u(0) = 0$ and is Lipschitz continuous. The integral representation in (7.8) has a unique solution x, so that the integral representation constitutes a function $H_u : D[0,\infty) \to D[0,\infty)$ mapping y into $x \equiv H_u(y)$. In addition, the function H_u is continuous, and if y is continuous, then so is x.

In our case, we set $u(x) = -\xi x$, to be able to write $\bar{X}_n = H_u\left(\bar{X}_n(0) + \bar{Z}_n\right)$. Since u is clearly Lipschitz continuous, the mapping H_u is indeed continuous. Let us rewrite (7.7), by observing

$$\mathbb{E}\bar{Z}_n(t) = \frac{1}{n}\Big(\mathbb{E}[N_b(nt)]\mathbb{E}[B] - \mathbb{E}[N_d(nt)]\mathbb{E}[D]\Big) = \lambda_b\mathbb{E}[B]t - \lambda_d\mathbb{E}[D]t, \tag{7.9}$$

where the expectation is taken with respect to the compound Poisson processes. Since $m = \lambda_b\mathbb{E}B - \lambda_d\mathbb{E}D$,

$$\bar{X}_n(t) = \bar{X}_n(0) - \xi\int_0^t \left(\bar{X}_n(s) - \frac{m}{\xi}\right)\mathrm{d}s + \bar{Y}_n(t), \tag{7.10}$$

where $\bar{Y}_n(t) := \bar{Z}_n(t) - mt$ is now a centered process. This allows us to state the next result.

PROPOSITION 7 (Fluid limit). *Let $\mathbb{E}[B], \mathbb{E}[D] < \infty$ and $\bar{X}_n(0) = X_n(0)/n \to q_0 \in \mathbb{R}$, as $n \to \infty$. Then for $n \to \infty$,*

$$\bar{X}_n \overset{d}{\Rightarrow} q, \tag{7.11}$$

where

$$q(t) = \frac{m}{\xi} + \left(q_0 - \frac{m}{\xi}\right)\mathrm{e}^{-\xi t}. \tag{7.12}$$

PROOF. First, we concentrate on the process $\bar{Y}_n$. Observe that, by the FLLN for renewal-reward processes, which follows from [30, Thm. 7.4.1], we have

$$\frac{1}{nt}\sum_{i=1}^{N_b(nt)} B_i \stackrel{d}{\Rightarrow} \lambda_b\mathbb{E}[B], \qquad \frac{1}{nt}\sum_{i=1}^{N_d(nt)} D_i \stackrel{d}{\Rightarrow} \lambda_d\mathbb{E}[D], \tag{7.13}$$

for $n \to \infty$ and for all $t > 0$. Hence, $\bar{Z}_n(t) \stackrel{d}{\Rightarrow} \lambda_b\mathbb{E}[B]t - \lambda_d\mathbb{E}[D]t = mt$. By definition of $\bar{Y}_n$ and the assumption of convergence of $\bar{X}_n(0)$, this implies

$$\bar{Y}_n + \bar{X}_n \stackrel{d}{\Rightarrow} q_0 \tag{7.14}$$

as $n \to \infty$. Next, note $\bar{X}_n = H_u\left(\bar{X}_n(0) + \bar{Z}_n\right) = H_u\left(\bar{X}_n(0) + \bar{Y}_n + mI\right)$, where I denotes the identity map, i.e. $I(t) \equiv t$ for all $t \geq 0$. Due to Lipschitz continuity of u, H_u constitutes a continuous mapping, and hence we can apply the Continuous Mapping Theorem (CMT), to find

$$\bar{X}_n = H_u\left(\bar{X}_n(0) + \bar{Y}_n + mt\right) \Rightarrow H_u\left(q_0 + mt\right) \equiv q, \tag{7.15}$$

for all $t \geq 0$, where $q(\cdot)$ is the solution of

$$\begin{aligned} q(t) &= q_0 + \int_0^t u(q(s))\,\mathrm{d}s = q_0 + mt - \xi\int_0^t q(s)\,\mathrm{d}s \\ &= q_0 - \xi\int_0^t \left(q(s) - \frac{m}{\xi}\right)\mathrm{d}s. \end{aligned}$$

The unique solution of this integral equation is given in (7.12). □

According to Proposition 7, the fluid limit approaches $\mathbb{E}[X] = m/\xi$ exponentially fast. To obtain an expression for the *diffusion limit* of the process, we analyze the fluctuations of the process around the fluid limit in (7.11), again by scaling the process in a proper manner. First, we subtract $q(t)$ on both sides of (7.10), and multiply by $\sqrt{n}$:

$$\sqrt{n}\left(\bar{X}_n(t) - q(t)\right) = \sqrt{n}\left(\bar{X}_n(0) - q_0\right) - \xi\int_0^t \sqrt{n}\left(\bar{X}_n(s) - q(s)\right)\mathrm{d}s + \sqrt{n}\,\bar{Y}_n(t). \tag{7.16}$$

Let $\hat{X}_n \equiv \sqrt{n}\left(\bar{X}_n - q\right)$ and $\hat{Y}_n \equiv \sqrt{n}\,\bar{Y}_n$, then this reduces to

$$\hat{X}_n(t) = \hat{X}_n(0) - \xi\int_0^t \hat{X}_n(s)\,\mathrm{d}s + \hat{Y}_n(t). \tag{7.17}$$

Again the process $\hat{Y}_n$ needs special attention.

LEMMA 1. *Let* $\mathbb{E}[B^2], \mathbb{E}[D^2] < \infty$. *Then* $\hat{Y}_n \overset{d}{\Rightarrow} \sigma W$ *as* $n \to \infty$, *where* $\sigma^2 := \lambda_b \mathbb{E}[B^2] + \lambda_d \mathbb{E}[D^2]$ *and* W *is a standard Brownian motion.*

PROOF. Recall that

$$\hat{Y}_n(t) \overset{d}{=} \sqrt{n}\left[\Big(\frac{1}{n}\sum_{i=1}^{N_b(nt)} B_i - \lambda_b \mathbb{E}[B]t\Big) - \Big(\frac{1}{n}\sum_{i=1}^{N_d(nt)} D_i - \lambda_d \mathbb{E}[D]t\Big)\right]. \tag{7.18}$$

By the Functional Central Limit Theorem (FCLT) for renewal-reward processes given in [30, Thm. 7.4.1], the process

$$\hat{Y}_n^b(t) = \sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{N_b(nt)} B_i - \lambda_b \mathbb{E}[B]t\Big), \tag{7.19}$$

converges weakly to $\sigma_b W_b$, where W_b is a standard Brownian motion, and

$$\sigma_b^2 = \lambda_b \operatorname{Var} B + \lambda_b (E[B])^2 = \lambda_b \mathbb{E}[B^2]. \tag{7.20}$$

Similarly $\hat{Y}_n^d \Rightarrow \sigma_d W_d$, with the obvious parameter switches and W_d is standard Brownian motion. Since the processes $\hat{Y}_n^b$ and $\hat{Y}_n^d$ are independent, so are their limits, and

$$\hat{Y}_n \Rightarrow \sqrt{\lambda_b \mathbb{E}[B^2]}\, W_b + \sqrt{\lambda_d \mathbb{E}[D^2]}\, W_d \overset{d}{=} \sqrt{\lambda_b \mathbb{E}[B^2] + \lambda_d \mathbb{E}[D^2]}\, W, \tag{7.21}$$

for $n \to \infty$ and W a standard Brownian motion. □

Now, we are ready to prove the diffusion counterpart of Proposition 7.

PROPOSITION 8 (Diffusion limit). *Let* $\mathbb{E}[B^2], \mathbb{E}[D^2] < \infty$. *If* $\hat{X}_n(0) \to \hat{X}(0)$, *then* $\hat{X}_n \Rightarrow \hat{X}$ *as* $n \Rightarrow \infty$, *where* $\hat{X}$ *satisfies the integral equation*

$$\hat{X}(t) = \hat{X}(0) - \xi \int_0^t \hat{X}(s)\,\mathrm{d}s + \sigma W(t). \tag{7.22}$$

In other words, $\hat{X}$ *is an Ornstein-Uhlenbeck diffusion process with infinitesimal mean* ξ *and infinitesimal variance* $\sigma^2 := \lambda_b \mathbb{E}[B^2] + \lambda_d \mathbb{E}[D^2]$.

PROOF. We again rely on the result that the mapping H_u as in the proof of Proposition 7 is continuous if u is Lipschitz continuous. Here, we set $u(x) = -\xi x$ which again clearly satisfies this condition. We have $\hat{X}_n \equiv H_u(\hat{X}_n(0) + \hat{Y}_n)$. From Lemma 1, we know

$$\hat{X}_n(0) + \hat{Y}_n \Rightarrow \hat{X}(0) + \sigma W, \tag{7.23}$$

for $n \to \infty$. As a consequence of the CMT, we conclude

$$\hat{X}_n = H_u\left(\hat{X}_n(0) + \hat{Y}_n\right) \Rightarrow H_u\left(\hat{X}(0) + \sigma W\right) \equiv \hat{X}, \tag{7.24}$$

where $\hat{X}$ solves (7.22). □

7.2. *Generalization for $\xi_b \neq \xi_d$.* We now sketch the scaling approach towards fluid and diffusion limits for the general case in which ξ_b may differ from ξ_d. In case $\xi_b \neq \xi_d$, the integral equation for $\bar{X}_n$ as in (7.7) becomes

(7.25)

$$\begin{aligned}\bar{X}_n(t) &= \bar{X}_n(0) + \int_0^t (-\xi_b \bar{X}_n^+(s) + \xi_d \bar{X}_n^-(s) - m)\,\mathrm{d}s + \bar{Y}_n(t) \\ &= \bar{X}_n(0) - \int_0^t (\left[\xi_b \mathbb{1}_{\{\bar{X}_n(s)\geq 0\}} + \xi_d \mathbb{1}_{\{\bar{X}_n(s)<0\}}\right] \bar{X}_n(s) + m)\,\mathrm{d}s + \bar{Y}_n(t),\end{aligned}$$

where $\bar{Y}_n(t)$ is defined as before. Note that $\hat{X}_n \equiv H_u(\bar{X}_n(0) + \bar{Y}_n)$, where we now have

$$u(x) = -\left[\xi_b \mathbb{1}_{\{x\geq 0\}} + \xi_d \mathbb{1}_{\{x<0\}}\right] x + m, \tag{7.26}$$

which is still Lipschitz continuous. Therefore, following the reasoning of the proof of Proposition 7, we obtain the fluid limit $\bar{X}_n \overset{d}{\Rightarrow} q$, where q is the solution of

$$q(t) = q_0 - \int_0^t (\left[\xi_b \mathbb{1}_{\{q(s)\geq 0\}} + \xi_d \mathbb{1}_{\{q(s)<0\}}\right] q(s) - m)\,\mathrm{d}s. \tag{7.27}$$

The solution to this integral equation is more elaborate than (7.11) and depends on the sign of m and q_0. Assuming $m \geq 0$, one can check that,

$$q(t) = \frac{m}{\xi_b} + \left(q_0 - \frac{m}{\xi_b}\right) e^{-\xi_b t}, \qquad \text{if } q_0 \geq 0, \tag{7.28}$$

$$q(t) = \begin{cases} \frac{m}{\xi_d} + \left(q_0 - \frac{m}{\xi_d}\right) e^{-\xi_d t}, & \text{if } 0 \leq t < t_d^*, \\ \frac{m}{\xi_b}\left(1 - e^{-\xi_b (t - t_d^*)}\right), & \text{if } t \geq t_d^*, \end{cases} \qquad \text{if } q_0 < 0, \tag{7.29}$$

where

$$t_d^* = -\frac{1}{\xi_d} \log\left(\frac{m/\xi_d}{m/\xi_d - q_0}\right). \tag{7.30}$$

If $m < 0$,

$$q(t) = \frac{m}{\xi_d} + \left(q_0 - \frac{m}{\xi_d}\right) e^{-\xi_d t}, \qquad \text{if } q_0 \leq 0, \tag{7.31}$$

$$q(t) = \begin{cases} \frac{m}{\xi_b} + \left(q_0 - \frac{m}{\xi_b}\right) e^{-\xi_b t}, & \text{if } 0 \le t < t_b^*, \\ \frac{m}{\xi_d}\left(1 - e^{-\xi_d (t - t_b^*)}\right), & \text{if } t \ge t_b^*, \end{cases} \qquad \text{if } q_0 > 0, \tag{7.32}$$

where

$$t_b^* = -\frac{1}{\xi_b} \log\left(\frac{m/\xi_b}{m/\xi_b - q_0}\right). \tag{7.33}$$

Note that the equilibrium of the fluid limit also depends on the sign of m:

$$\lim_{t\to\infty} q(t) = \begin{cases} m/\xi_b, & \text{if } m \ge 0, \\ m/\xi_d, & \text{if } m < 0. \end{cases} \tag{7.34}$$

In the remainder, without loss of generality $m \ge 0$. Furthermore, set $q_0 = m/\xi_b$ so that $q \equiv m/\xi_b$. Subtracting $q(t)$ on both sides of (7.25) yields,

(7.35)

$$\begin{aligned} \left(\bar{X}_n(t) - q(t)\right) &= \left(\bar{X}_n(0) - q_0\right) \\ &\quad - \int_0^t \left\{ \left[\xi_b \mathbb{1}_{\{\bar{X}_n(s)\ge 0\}} + \xi_d \mathbb{1}_{\{\bar{X}_n(s)<0\}}\right] \bar{X}_n(s) - \xi_b\, q(s)\right\} \mathrm{d}s \\ &\quad + \bar{Y}_n(t) \\ &= \left(\bar{X}_n(0) - q_0\right) - \int_0^t \xi_b \left(\bar{X}_n(s) - q(s)\right) \mathrm{d}s \qquad (7.36) \\ &\quad + \int_0^t \mathbb{1}_{\{\bar{X}_n(s)<0\}} (\xi_b - \xi_d) \bar{X}_n(s)\, \mathrm{d}s. \end{aligned}$$

Let $\hat{X}_n(t) = \sqrt{n}\left(\bar{X}_n(t) - q(t)\right)$. Then

(7.37)

$$\hat{X}_n(t) = \hat{X}_n(0) - \xi_b \int_0^t \hat{X}_n(s)\, \mathrm{d}s + \int_0^t \mathbb{1}_{\{\bar{X}_n(s)<0\}} (\xi_b - \xi_d) \bar{X}_n(s)\, \mathrm{d}s + \hat{Y}_n(t)$$

Now, we argue non-rigorously that the one-but-last term vanishes as $n \to \infty$. Namely, by defining the function $G : D[0,\infty) \to D[0,\infty)$ by the integration operator:

$$G(u) = \int_0^t \mathbb{1}_{\{u(s)<0\}} (\xi_b - \xi_d) u(s)\, \mathrm{d}s, \tag{7.38}$$

this term can be expressed as $G(\bar{X}_n)$. Hence by the fact that $\hat{X}_n \overset{d}{\Rightarrow} m/\xi_b$ and the CMT we see $G(\hat{X}_n) \Rightarrow 0$.

Under this claim, we deduce by the approach of Proposition 8, that if $\hat{X}_n \Rightarrow \hat{X}$ for $n \to \infty$, then $\hat{X}$ satisfies the stochastic integral equation

$$\hat{X}(t) = \hat{X}(0) - \xi_b \int_0^t \hat{X}(s)\,\mathrm{d}s + \sigma W(t), \tag{7.39}$$

which implies that $\hat{X}$ is an Ornstein-Uhlenbeck process with infinitesimal mean ξ_b and variance $\sigma^2 := \lambda_b \mathbb{E}[B^2] + \lambda_d \mathbb{E}[D^2]$.

The result that the scaled process converges to an Ornstein-Uhlenbeck process can be intuitively justified by the so-called *mean-reverting* behavior of the original process. That is, the further the process is away from its mean, the greater the drift towards that equilibrium. This is the defining feature of the OU diffusion process. The decay rates ξ_b and ξ_d are responsible for the original process being 'forced' towards 0 and therefore the similarities should not be surprising. However, note that in the diffusion limit $\hat{X}_n$ has drift ξ_b (cq. ξ_d) towards m/ξ_b (cq. m/ξ_d), if $m > 0$ (cq. < 0) at *any* position of the process. This implies that if $X_n \in (0, nm/\xi_b)$, it has an upward drift, which is at first sight counter-intuitive. However, we can argue that in case $X_n(t) = v \in (0, nm/\xi_b)$, the mean upward drift of the process X_n equals $n\lambda_b \mathbb{E}[B]$, and the mean downward drift equals $n\lambda_d \mathbb{E}[D] + \xi_b v$, since $v > 0$. Rewrite $v = nm/\xi_b - w\sqrt{n}$ for some $w \in (0, \sqrt{n}m/\xi_b)$. Then, the mean net drift equals

$$n\lambda_b \mathbb{E}[B] - n\lambda_d \mathbb{E}[D] - \xi_b \left(\frac{nm}{\xi_b} - w\sqrt{n} \right) = \xi_b w \sqrt{n} > 0,$$

which explains both the sign and magnitude of the drift factor in the scaled process.

7.3. *Related literature.* The Ornstein-Uhlenbeck process is a diffusion process that often arises as the limit of a sequence of stochastic systems, in which the system size tends to infinity. Particularly in queueing settings with mean reverting behavior, the OU process appears in so-called heavy traffic, i.e. the arrival rate grows without bound. We mention a couple of models that exhibit limiting behavior that is similar to ours.

First, it is well-known that the properly normalized $M/M/\infty$ queue length process converges weakly to a OU process as the arrival rate tends to infinity, see e.g. [30, Sec. 10.3]. This limiting behavior continues to hold in case the queueing process is modulated by a Markovian background process, see [2].

Another well-known queueing model in which a (piecewise) OU process appears in the limit is the multi-server queue with abandonments. For the $M/M/s+M$ queue, where $+M$ denotes the exponentially distributed patience of customers, Garnett et al. [16] showed that in the Halfin-Whitt regime, the queue length process, centered and scaled around the number of servers s, approaches a hybrid OU process, of which the drift parameter depends on the current state: If the queue length is larger (cq. smaller) than zero, then the drift is governed by the abandonment rate (cq. service rate). Dai et al. [13] find a similar piecewise diffusion process under more general assumptions on the model primitives.

For the single-server queue with abandoning customers, Ward and Glynn [28, 29] showed that in conventional heavy traffic, the queue length process converges to a OU process with reflecting barrier 0.

Since we in our setting assumed both demand impatience and perishability of inventory (which can be seen as a kind of impatience as well), it should not come as a surprise that we also find our limiting process to be a OU process. Observe however that in our model, unless $m=0$, we find a OU process with constant, rather than piecewise, parameters, and no reflection barrier, since our (scaled) inventory process can go both positive and negative.

Last, we mention that there is a connection between our blood inventory process and the work of Reed and Zwart [23]. Rather than looking at the OU process as the limit of a sequence of stochastic processes, Reed and Zwart [23] study a stochastic differential equation that is closely related to Equation (7.2), in the sense that the process has a different (constant) drift term in the upper and lower half plane. Under the assumption that the input process is a Lévy process with only one-sided jumps, they develop a methodology to derive the invariant distribution of the solution of the SDE. Unfortunately, the input in our scenario exhibits both positive and negative jumps, which prevents us from applying their results directly to (7.2).

8. Numerical evaluation.

8.1. *Approximation scheme.* The asymptotic results of the previous section regarding the fluid and diffusion limits allude to the fact that for large arrival rates, the normalized inventory process $\{\hat{X}_n(t)\,|\,t\geq 0\}$, resembles that of the Ornstein-Uhlenbeck process. Indeed, the sample paths of the scaled process $\bar{X}_n$ for increasing values of n in Figures 4 and 5 show that the mean-reverting behavior around m/ξ^*, that is typical of OU processes, kicks in rather quickly. Moreover, the fluid limits $q(t)$ as presented by Proposition 7 and (7.29)–(7.32) predict the mean well for both $\xi_b=\xi_d$ and $\xi_b\neq\xi_d$.

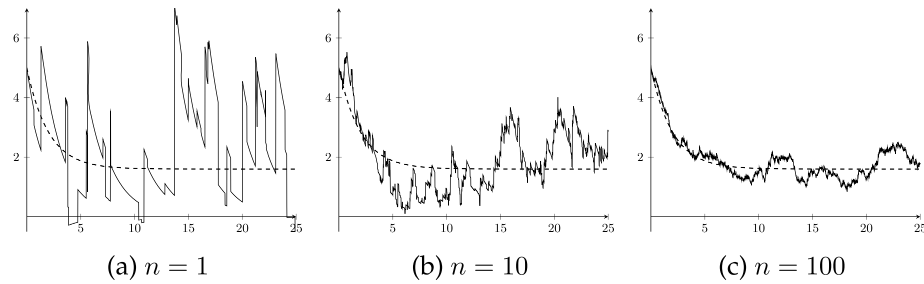


FIG 4. *Sample paths of the scaled inventory process* $\bar{X}_n(t) = X_n(t)/n$ *with* $\bar{X}_n(0) = 5$, $\lambda_b = 1.2$, $\lambda_d = 1$, $\xi_b = \xi_d = 0.5$ *and* $\mu_b = 0.5$ *and* $\mu_d = 1$. *The fluid limit is depicted by the dashed line.*

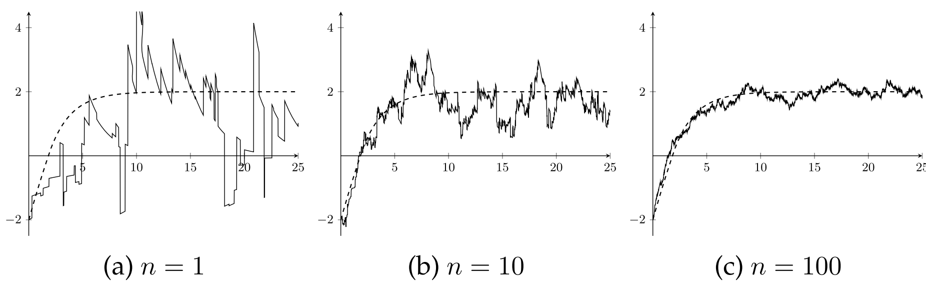


FIG 5. *Sample paths of the scaled inventory process* $\bar{X}_n(t) = X_n(t)/n$ *with* $\bar{X}_n(0) = -2$, $\lambda_b = 2$, $\lambda_d = 1$, $\xi_b = 0.5, \xi_d = 0.1$ *and* $\mu_b = 1$ *and* $\mu_d = 1$. *The fluid limit is depicted by the dashed line.*

Furthermore, we observe that steady state is attained fairly quickly. This is suggestive of the claim that the steady-state distribution of the normalized process $\hat{X}_n$ is well-described by the steady-state distribution of the OU process $\hat{X}$. Since the OU process with mean 0, infinitesimal variance σ^2 and drift ξ^* is known to be normally distributed with mean 0 and variance $\sigma^2/2\xi^*$ in steady-state, this leads to a simpler approximation scheme based on the first two moments of B and D only. In non-rigorous mathematical terms, we use the approximation that

$$\hat{X}_n = \frac{X_n - nm/\xi^*}{\sqrt{n}} \overset{d}{\approx} Z^*, \tag{8.1}$$

where Z^* is a normally distributed random variable with mean 0 and variance $\sigma^2/2\xi^*$.

Note that justification of the conjecture that the normal approximation is indeed an asymptotically correct approximation for systems with large arrival rates requires proof that the interchange-of-limits between $t \to \infty$ and $n \to \infty$ is indeed valid. Rather than going into the technical details, we provide in the remainder of this section numerical evidence that this interchange indeed holds, and that the normal approximation is able to capture characteristics of processes with exponential jumps as well as generally distributed jumps.

8.2. *Distribution functions.* Since we obtained an explicit expression for the steady-state density function of the net inventory process X in case B and D are exponential, see Theorem 5, we will exploit this formula for numerical comparison to the normal approximation arising from the OU process.

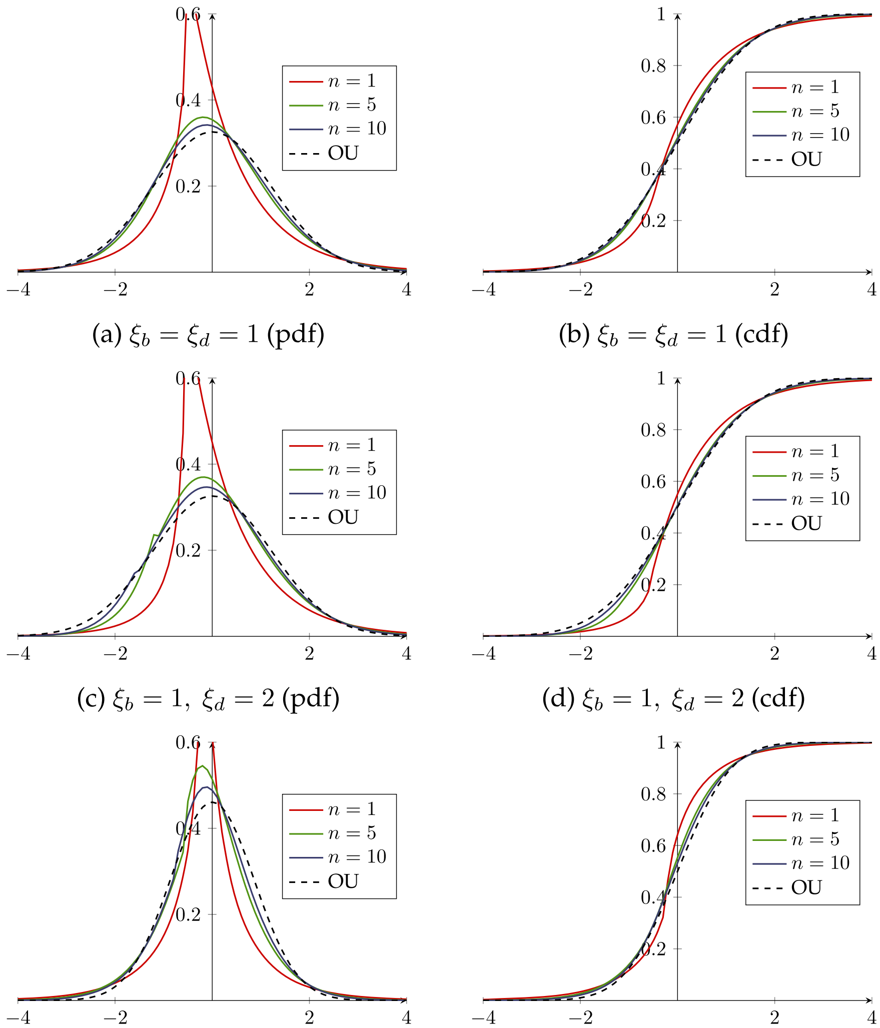
Let $h(\cdot)$ as in Theorem 5 be the pdf of X with parameters λ_b, λ_d, μ_b, μ_d, ξ_b and ξ_d, and the corresponding cdf H, defined as $H(v) = \int_{-\infty}^{v} h(x)\mathrm{d}x$. We denote by h_n and H_n the pdf and cdf, respectively, of the inventory process X_n with arrival rates $n\lambda_b$ and $n\lambda_d$, and the remaining parameters unchanged. Then, the pdf and cdf of the normalized process are given by $\hat{h}_n(v) = \sqrt{n}\, h_n(v_n)$ and $\hat{H}_n(v) = H_n(v_n)$, respectively, with $v_n = nm/\xi^* + v\sqrt{n}$ for all $v \in \mathbb{R}$. By the normal approximation scheme, we expect

$$\hat{h}_n(v) \approx \frac{\sqrt{2\xi^*}}{\sigma}\, \varphi\left(\frac{\sqrt{2\xi^*}}{\sigma} v\right), \quad \text{and} \quad \hat{H}_n(v) \approx \Phi\left(\frac{\sqrt{2\xi^*}}{\sigma} v\right). \tag{8.2}$$

We perform this numerical comparison of probability functions in Figure 6 for three cases: $\xi_b = \xi_d$, $\xi_b > \xi_d$ and $\xi_b < \xi_d$.

From Figure 6, in which $m = 1$, so that $\xi^* = \xi_b$, the convergence of the pdf and cdf is evident. For $n = 10$, the distribution functions of the scaled processes are almost aligned with the normal distribution already. For $\xi_b = \xi_d$, the convergence is fastest. This can be explained by observing that in cases where $\xi_b \neq \xi_d$, the parameter ξ_d still plays a role in pre-limit systems, whereas it does not appear in the normal limit. In the cases where $\xi_b \neq \xi_d$ we furthermore see that the functions are not smooth around $v_n = 0$ or $v^* = -\sqrt{n}m/\xi^*$, which is the zero-inventory level in the original (unscaled) process. As n increases, this point of irregularity goes to $-\infty$ and therefore disappears.

8.3. *Approximations to performance metrics.* The plots in the previous section indicate that the normal approximation gives simple yet accurate approximations to the stationary distribution of the inventory process. We now assess if this also translates to the performance measures. Again, we



(a) $\xi_b = \xi_d = 1$ (pdf) (b) $\xi_b = \xi_d = 1$ (cdf)

(c) $\xi_b = 1,\ \xi_d = 2$ (pdf) (d) $\xi_b = 1,\ \xi_d = 2$ (cdf)

(e) $\xi_b = 2,\ \xi_d = 1$ (pdf) (f) $\xi_b = 2,\ \xi_d = 1$ (cdf)

FIG 6. *Steady-state density (left) and distribution (right) function of the normalized inventory process $\hat{X}_n$ for $n = 1,\ 5$ and 10 with $\lambda_b = 1,\ \lambda_d = 0.5$ and $\mu_b = \mu_d = 1$, and of the OU process.*

choose to fix the parameters λ_b and λ_d, and evaluate the system with arrival rates $n\lambda_b$ and $n\lambda_d$ for increasing n. First, the normal approximation in (8.1) yields the following approximation for the expected inventory level:

$$\mathbb{E}[X_n] \approx \frac{nm}{\xi^*} = \frac{n(\lambda_b \mathbb{E}[B] - \lambda_d \mathbb{E}[D])}{\xi^*}. \tag{8.3}$$

For the probability of negative inventory, we have

$$\pi_d = \mathbb{P}(X_n < 0) \approx \mathbb{P}\left(Z^* < -\sqrt{n}\, m/\xi^*\right) = \Phi\left(-\sqrt{n/2\xi^*}\, m/\sigma\right). \tag{8.4}$$

Last, the probability of demand being satisfied immediately is approximately

$$\begin{aligned} \mathbb{P}(\text{demand satisfied}) &= \mathbb{P}(X_n > D) \\ &\approx 1 - \int_0^\infty \Phi\left(-\frac{\sqrt{2\xi^*}}{\sigma}\, \frac{x - nm/\xi^*}{\sqrt{n}}\right) \mathrm{d}F_d(x). \end{aligned} \tag{8.5}$$

REMARK 6. *Note that if* λ_b *and* λ_d *are large themselves, the parameter* n *can be eliminated from* (8.3)–(8.5)*, so that*

$$\mathbb{E}[X] \approx \frac{m}{\xi^*}, \qquad \pi_d \approx \Phi\left(-m/(\sigma\sqrt{2\xi^*})\right),$$
$$\mathbb{P}(\text{demand satisfied}) \approx 1 - \int_0^\infty \Phi\left(-\sqrt{2\xi^*}\, \frac{x - m/\xi^*}{\sigma}\right) \mathrm{d}F_d(x),$$

where $m = \lambda_b \mathbb{E}[B] - \lambda_d \mathbb{E}[D]$ *and* $\sigma^2 = \lambda_b \mathbb{E}[B^2] + \lambda_d \mathbb{E}[D^2]$.

We will now test these approximations under various assumptions on the distribution of B and D. In Tables 1–3 we compare the values obtained through the normal approximation against the true values obtained through numerical evaluation (for exponential jump sizes only) and simulation. All simulation results are accurate up to a 95% confidence interval of width 10^{-4}. We set $\lambda_b = 1$ and $\lambda_d = 0.5$ and let the mean jump sizes be equal to 1, i.e. $\mathbb{E}[B] = 1$ and $\mathbb{E}[B] = 1$ in all numerical experiments. In Table 1, we let the jump sizes be deterministic, so that $\operatorname{Var} B = \operatorname{Var} D = 0$. Table 2 shows the results in case of exponential jump sizes, so that $\operatorname{Var} B = \operatorname{Var} D = 1$. Last, in Table 3 we investigate the quality of the approximation for jump sizes that follow a $\text{Gamma}(0.25, 0.25)$ distribution, yielding $\operatorname{Var} B = \operatorname{Var} D = 4$. With this set-up we cover jump distributions of increasing variance, so that we are able to study the impact of increased variability on the accuracy of the approximations. Moreover, we investigate the influence of the decay parameters ξ_b and ξ_d by considering the scenarios $\xi_b = \xi_d$, $\xi_b < \xi_d$ and $\xi_b > \xi_d$.

We make a couple of observations based on the numbers in Tables 1–3. First, we see that the approximation for the mean blood inventory level $\mathbb{E}[X_n]$ is exact if $\xi_b = \xi_d$, see Proposition 3. This obviously does not extend to π_d and $\mathbb{P}(\text{demand satisfied})$, since these performance measures are based on the entire distribution of X_n rather than the mean. Nonetheless, the normal approximation appears to be most accurate in the case $\xi_b = \xi_d$. We

TABLE 1
Accuracy of diffusion approximation for the mean blood inventory level $\mathbb{E}[X_n]$, the probability of negative inventory π_d and the probability of demand being fully satisfied immediately $\mathbb{P}$(dem. sat.), with arrival rates $n\lambda_b = n$ and $n\lambda_d = 0.5n$ and deterministic jump sizes, $B \equiv 1$ and $D \equiv 1$.

	$\mathbb{E}[X_n]$		π_d		$\mathbb{P}$(dem.sat.)	
n	Sim.	(8.3)	Sim.	(8.4)	Sim.	(8.5)
1	0.500	0.500	0.2702	0.2819	0.2598	0.2819
2	1.000	1.000	0.2014	0.2071	0.4859	0.5000
5	2.500	2.500	0.0943	0.0984	0.7814	0.7807
10	5.000	5.000	0.0316	0.0339	0.9306	0.9279
20	10.000	10.000	0.0043	0.0049	0.9908	0.9899
50	25.000	25.000	0.0000	0.0000	1.0000	1.0000

(a) $\xi_b = 1$, $\xi_d = 1$.

	$\mathbb{E}[X_n]$		π_d		$\mathbb{P}$(dem.sat.)	
n	Sim.	(8.3)	Sim.	(8.4)	Sim.	(8.5)
1	0.584	0.500	0.2522	0.2819	0.2712	0.2819
2	1.086	1.000	0.1809	0.2071	0.5020	0.5000
5	2.558	2.500	0.0837	0.0984	0.7911	0.7807
10	5.024	5.000	0.0286	0.0339	0.9335	0.9279
20	10.006	10.000	0.0040	0.0049	0.9912	0.9899
50	25.000	25.000	0.0000	0.0000	1.0000	1.0000

(b) $\xi_b = 1$, $\xi_d = 2$.

	$\mathbb{E}[X_n]$		π_d		$\mathbb{P}$(dem.sat.)	
n	Sim.	(8.3)	Sim.	(8.4)	Sim.	(8.5)
1	0.158	0.250	0.3308	0.3415	0.1006	0.1103
2	0.397	0.500	0.2973	0.2819	0.2465	0.2819
5	1.164	1.250	0.1952	0.1807	0.5482	0.5724
10	2.447	2.500	0.1036	0.0984	0.7729	0.7807
20	4.980	5.000	0.0340	0.0339	0.9283	0.9279
50	12.497	12.500	0.0017	0.0019	0.9964	0.9960

(c) $\xi_b = 2$, $\xi_d = 1$.

may explain this by observing that in the approximations (8.3)–(8.5), only ξ^* appears. In our setting, we have $m = \lambda_b - \lambda_d = 0.5$, so that $\xi^* = \xi_b$. If $\xi_b \neq \xi_d$, then the value of ξ_d plays a role in pre-limit systems, which induces inaccuracies in the approximation of performance measures. In case $\xi_b = \xi_d$, we have $\xi^* = \xi_b = \xi_d$, so that this discrepancy is overcome.

Moreover, since $m > 0$, we see that $\pi_d \to 0$ and $\mathbb{P}$(demand satisfied) $\to 1$ as n increases. This is due to the observation that as n grows large, the

TABLE 2
Accuracy of diffusion approximation for the mean blood inventory level $\mathbb{E}[X_n]$, the probability of negative inventory π_d and the probability of demand being fully satisfied immediately $\mathbb{P}(\text{dem. sat.})$, with arrival rates $n\lambda_b = n$ and $n\lambda_d = 0.5n$ and exponentially distributed jump sizes, $B \sim \exp(1)$ and $D \sim \exp(1)$.

	$\mathbb{E}[X_n]$		π_d		$\mathbb{P}(\text{dem.sat.})$	
n	Exact	(8.3)	Exact	(8.4)	Exact	(8.5)
1	0.500	0.500	0.2929	0.3415	0.3536	0.3925
2	1.000	1.000	0.2500	0.2819	0.5000	0.5135
5	2.500	2.500	0.1642	0.1807	0.7062	0.7009
10	5.000	5.000	0.0898	0.0984	0.8491	0.8418
20	10.000	10.000	0.0307	0.0339	0.9506	0.9467
50	25.000	25.000	0.0017	0.0019	0.9974	0.9970

(a) $\xi_b = 1$, $\xi_d = 1$.

	$\mathbb{E}[X_n]$		π_d		$\mathbb{P}(\text{dem.sat.})$	
n	Exact	(8.3)	Exact	(8.4)	Exact	(8.5)
1	0.621	0.500	0.2589	0.3415	0.3705	0.3925
2	1.153	1.000	0.2164	0.2819	0.5224	0.5135
5	2.656	2.500	0.1414	0.1807	0.7254	0.7009
10	5.113	5.000	0.0784	0.0984	0.8598	0.8418
20	10.050	10.000	0.0275	0.0339	0.9538	0.9467
50	25.004	25.000	0.0016	0.0019	0.9975	0.9970

(b) $\xi_b = 1$, $\xi_d = 2$.

	$\mathbb{E}[X_n]$		π_d		$\mathbb{P}(\text{dem.sat.})$	
n	Exact	(8.3)	Exact	(8.4)	Exact	(8.5)
1	0.125	0.250	0.3548	0.3864	0.2168	0.2942
2	0.333	0.500	0.3333	0.3415	0.3333	0.3925
5	1.059	1.250	0.2647	0.2593	0.5264	0.5570
10	2.333	2.500	0.1856	0.1807	0.6881	0.7009
20	4.893	5.000	0.0995	0.0984	0.8400	0.8418
50	12.475	12.500	0.0198	0.0206	0.9692	0.9678

(c) $\xi_b = 2$, $\xi_d = 1$.

inventory process concentrates around the level nm with fluctuations of order $\sqrt{n}$, so that the process stays away from level zero, see Figure 4. The approximations (8.4)–(8.5) adequately capture this convergence.

As expected, the accuracy of the approximations increases with n. Moreover, increased variability in the jump distributions appears to cause a decrease in accuracy. However, for all cases considered in Tables 1–3, the normal approximations (8.3)–(8.5) seem to yield relatively sharp estimates for

TABLE 3

Accuracy of diffusion approximation for the mean blood inventory level $\mathbb{E}[X_n]$, the probability of negative inventory π_d and the probability of demand being fully satisfied immediately $\mathbb{P}$(dem. sat.), with arrival rates $n\lambda_b = n$ and $n\lambda_d = 0.5n$ and Gamma distributed jump sizes, $B \sim$ Gamma$(0.25, 0.25)$ and $D \sim$ Gamma$(0.25, 0.25)$.

	$\mathbb{E}[X_n]$		π_d		$\mathbb{P}$(dem.sat.)	
n	Sim.	(8.3)	Sim.	(8.4)	Sim.	(8.5)
1	0.500	0.500	0.3118	0.3981	0.4412	0.4636
2	1.000	1.000	0.2894	0.3575	0.5343	0.5288
5	2.500	2.500	0.2375	0.2819	0.6590	0.6381
10	5.000	5.000	0.1785	0.2071	0.7592	0.7385
20	10.000	10.000	0.1090	0.1241	0.8593	0.8454
50	25.000	25.000	0.0303	0.0339	0.9624	0.9583

(a) $\xi_b = 1$, $\xi_d = 1$.

	$\mathbb{E}[X_n]$		π_d		$\mathbb{P}$(dem.sat.)	
n	Sim.	(8.3)	Sim.	(8.4)	Sim.	(8.5)
1	0.667	0.500	0.2695	0.3981	0.4636	0.4636
2	1.253	1.000	0.2469	0.3575	0.5632	0.5288
5	2.863	2.500	0.2009	0.2819	0.6895	0.6381
10	5.385	5.000	0.1518	0.2071	0.7834	0.7385
20	10.328	10.000	0.0938	0.1241	0.8739	0.8454
50	25.124	25.000	0.0269	0.0339	0.9658	0.9583

(b) $\xi_b = 1$, $\xi_d = 2$.

	$\mathbb{E}[X_n]$		π_d		$\mathbb{P}$(dem.sat.)	
n	Sim.	(8.3)	Sim.	(8.4)	Sim.	(8.5)
1	0.081	0.250	0.3694	0.4276	0.3270	0.4104
2	0.238	0.500	0.3593	0.3981	0.4137	0.4636
5	0.857	1.250	0.3237	0.3415	0.5311	0.5528
10	2.045	2.500	0.2739	0.2819	0.6282	0.6381
20	4.568	5.000	0.2039	0.2071	0.7361	0.7385
50	12.231	12.500	0.0966	0.0984	0.8797	0.8779

(c) $\xi_b = 2$, $\xi_d = 1$.

the relevant performance measures under various assumptions on the distributions of the jump sizes.

9. Conclusions and suggestions for further research. In this paper we have studied a stochastic model for a blood bank. We have presented a global approach to the model in its full generality, and we have obtained very detailed exact expressions for the densities of amount of inventory and amount of demand (shortage) in special cases (exponential amounts of donated and requested blood; and either $\alpha_b = \alpha_d = 0$ or $\xi_b = \xi_d = 0$). Moreover, we have shown how an appropriate scaling, for the model in full generality, leads to an Ornstein-Uhlenbeck diffusion process, which can be used as a tool to obtain simple yet accurate approximations for some key performance measures.

Our model is a two-sided model, in the sense that we simultaneously consider the amount of blood in inventory and the amount of demand (shortage), one of the two at any time being zero. Such two-sided processes arise in many different settings, and thus are of considerable interest. The present setting is reminiscent of an organ transplantation problem, where there is either a queue of persons waiting to receive an organ, or a queue of donor organs. The perishability/impatience aspect features there, too [9]. A quite different setting is that of insurance risk. We refer to Albrecher and Lautscham [1] who extend the classical Cramér-Lundberg insurance risk model by allowing the capital of an insurance company to become negative – a situation that is usually indicated by "ruin" in the insurance literature. Their process thus becomes two-sided. The capital might become positive again; however, at a rate $\omega(x)$ when the capital has a negative value $-x$, bankruptcy is declared and the process ends. Interestingly, similar special functions (like hypergeometric functions) play a role in [1] and in the present study.

Our results are restricted to one type of blood. It would be very interesting to extend the analysis to multiple types of blood. Another important extension would be to use our results to facilitate the decision process that is faced by the CBB on a daily basis: Which amounts of blood, and of which types, should today be sent to the local blood banks (hospitals)? Knowing that, e.g., blood types O^-, A^-, B^-, AB^- can satisfy the corresponding + type (but not vice versa), one may try to optimize the blood allocation process on the basis of actual amounts of blood present.

Finally, we mention a significant open research question regarding the process limits that we derived in Section 7, of which the steady-state distributions were used to approximate steady-state performance measures in pre-limit systems. As we pointed out earlier, the justification that the steady-state distribution of the scaled inventory process indeed converges to the steady-state distribution of the fluid (cq. diffusion) limit requires a rigorous argument why the order of limits $n \to \infty$ and $t \to \infty$ may be interchanged. Proving interchange-of-limits statements typically raises many technical challenges, see e.g. [12, 14, 18, 15] for works tackling this issue in the context of queues in heavy traffic. The usual approach is to prove tightness of the sequence of steady-state distributions of pre-limit, followed by applying Prokhorov's theorem, see e.g. [7, Sec. 1.5]. For our model, such an approach seems to be straightforward for the fluid scaling, since our inventory process can be upper (cq. lower) bounded by a shot-noise process with only positive (cq. negative) jumps. Of the latter, the steady-state behavior is known. This allows us to derive a uniform bound on the absolute mean of the stationary fluid-scaled process, which gives tightness. The final step uses the deterministic nature of the differential equation governing the dynamics of the fluid limit, by which the steady-state distribution must be unique. For the diffusion-scaled process, the steps towards proving the interchange-of-limits are not obvious and hence this needs further investigation. Our numerical results for various jump size distributions, however, support the conjecture that this interchange is indeed valid.

APPENDIX A: TRANSFORMATION OF INTEGRAL EQUATION (4.1)

In this appendix we show how integral equation (4.1) can be transformed into a second-order differential equation, in the case of exponential $F_b(\cdot)$ and $F_d(\cdot)$. Differentiate (4.1) w.r.t. v:

$$\text{(A.1)} \qquad \begin{aligned} \lambda_d f(v) - \mu_d \left[\lambda_d \int_0^v f(y)\mathrm{e}^{-\mu_d(v-y)}\mathrm{d}y + \lambda_d \int_0^\infty g(y)\mathrm{e}^{-\mu_d(v+y)}\mathrm{d}y\right] \\ = -\lambda_b f(v) + \lambda_b\mu_b \int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y + \xi_d f(v) + \xi_d v f'(v). \end{aligned}$$

Using (4.1) once more, now to replace the term between square brackets in (A.1), we get:

$$\text{(A.2)} \qquad \begin{aligned} \xi_d v f'(v) &= (\lambda_d + \lambda_b - \xi_d) f(v) \\ &\quad - \mu_d \left(\lambda_b \int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y + \xi_d v f(v)\right) \\ &\quad - \mu_b\lambda_b \int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y, \end{aligned}$$

and once more differentiating w.r.t. v then gives:

(A.3)

$$\begin{aligned} &\xi_d v f''(v) + \xi_d f'(v) - (\lambda_d + \lambda_b - \xi_d - \mu_d\xi_d v) f'(v) \\ &\quad = -\mu_d\xi_d f(v) + (\mu_b + \mu_d)\lambda_b f(v) - \mu_b(\mu_b + \mu_d)\lambda_b \int_v^\infty f(y)\mathrm{e}^{-\mu_b(y-v)}\mathrm{d}y. \end{aligned}$$

The integral that appears in (A.3) can be eliminated by using (A.2), and we thus finally obtain the following second order homogeneous differential equation:

$$\text{(A.4)}\qquad \xi_d v f''(v) + (2\xi_d - \lambda_d - \lambda_b + \mu_d\xi_d v - \mu_b\xi_d v)\, f'(v) + (\mu_d\xi_d - \mu_b\xi_d - \mu_d\lambda_b + \mu_b\lambda_d - \mu_b\mu_d\xi_d v)\, f(v) = 0.$$

APPENDIX B: PROOF OF PROPOSITION 4

In the proof, we concentrate on the derivation of $f(v)$, which is the solution to

$$\text{(B.1)}\qquad \xi_d v f''(v) + (2\xi_d - \lambda_d - \lambda_b + \mu_d\xi_d v - \mu_b\xi_d v)\, f'(v) + (\mu_d\xi_d - \mu_b\xi_d - \mu_d\lambda_b + \mu_b\lambda_d - \mu_b\mu_d\xi_d v)\, f(v) = 0$$

The expression for $g(v)$ follows directly from exchanging λ_b with λ_d, μ_b with μ_d, ξ_b with ξ_d, and π_b with π_d in $f(v)$. We rewrite (B.1) as follows:

$$\text{(B.2)}\qquad v f''(v) + (A + Bv) f'(v) + (C + Dv) f(v) = 0,$$

where

$$A = 2 - \frac{\lambda_b + \lambda_d}{\xi_d}, \quad B = \mu_d - \mu_b, \quad C = \mu_d - \mu_b + \frac{\lambda_d\mu_b - \lambda_b\mu_d}{\xi_d}, \quad D = -\mu_b\mu_d.$$

Note that we divided both sides of equation (4.3) by ξ_d here. We will try to transform the differential equation into one for which the solution is easily derived. In order to do so, we first guess f to be of the form $f(v) = e^{\beta v} h(v)$, where β is a constant and h another real-valued function. Substituting this into (B.2) gives

$$\text{(B.3)}\qquad v h''(v) + \left[(2\beta + B)v + A\right] h'(v) + \left[(\beta^2 + B\beta + D)v + A\beta + C\right] h(v) = 0.$$

Next, we would like to choose β such that $\beta^2 + B\beta + D = 0$, that is

$$\text{(B.4)}\qquad \beta = \frac{-B \pm \sqrt{B^2 - 4D}}{2},$$

which equals either $-\mu_d$ or μ_b. Since the solution of (B.2) we are looking for is a density, and necessarily $f(v) = e^{\beta v} h(v) \to 0$ as $v \to \infty$, we set β equal to the negative root $-\mu_d$. Lastly, we apply a change of variable, $x = \delta v$, and $h(v) = w(x)$, so that (B.3) is transformed into

$$\text{(B.5)}\qquad x w''(x) + \left[(2\beta + B)\delta^{-1} x + A\right] w'(x) + \delta^{-1}\left[A\beta + C\right] w(x) = 0.$$

By choosing $(2\beta + B)\delta^{-1} = -1$, i.e.

$$\text{(B.6)}\qquad \delta = -(2\beta + B) = \mu_b + \mu_d,$$

we obtain

$$xw''(x) + [A - x]w'(x) + \delta^{-1}[A\beta + C]\, w(x) = 0, \tag{B.7}$$

which is known as Kummer's equation, $xw''(x) + (b - x)w'(x) - aw(x) = 0$, see [25], with parameters

$$a = -\delta^{-1}[A\beta + C] = 1 - \frac{\lambda_d}{\xi_d},$$
$$b = A = 2 - \frac{\lambda_b + \lambda_d}{\xi_d}.$$

Kummer's equation has two linearly independent solutions, namely $w(x) = M(a, b, x)$, where M is Kummer's hypergeometric function, also denoted by ${}_1F_1(a, b, x)$, and $U(a, b, x)$, Tricomi's hypergeometric function. These are defined as, see [25, Eq. (1.3.1)],

$$M(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} x^n, \tag{B.8}$$

$$U(a, b, x) = \frac{\Gamma(b-1)}{\Gamma(1+a-b)}\, M(a, b, x) + \frac{\Gamma(b-1)}{\Gamma(a)}\, x^{1-b}\, M(1 + a - b, 2 - b, x), \tag{B.9}$$

where $(.)_n$ is the Pochhammer symbol, which is used to represent $(y)_n = y \cdot (y + 1) \cdot ... \cdot (y + n - 1)$. We can therefore deduce that $f(v)$ is of the form

$$e^{\beta v}\left[c_1\, M(a, b, \delta v) + c_2\, U(a, b, \delta v)\right], \tag{B.10}$$

or

$$e^{-\mu_d v}\Big[c_1 M\left(1 - \tfrac{\lambda_d}{\xi_d}, 2 - \tfrac{\lambda_b+\lambda_d}{\xi_d}, (\mu_b + \mu_d)v\right) + c_2 U\left(1 - \tfrac{\lambda_d}{\xi_d}, 2 - \tfrac{\lambda_b+\lambda_d}{\xi_d}, (\mu_b + \mu_d)v\right)\Big], \tag{B.11}$$

where c_1 and c_2 are constants. From [25, p. 60], we have

$$M(a, b, x) \sim \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}, \qquad \text{as } x \to \infty. \tag{B.12}$$

Hence,

$$e^{-\mu_d v} M\left(1 - \frac{\lambda_d}{\xi_d}, 2 - \frac{\lambda_b + \lambda_d}{\xi_d}, (\mu_b + \mu_d)v\right) \tag{B.13}$$

$$\sim \frac{\Gamma(2-\frac{\lambda_b+\lambda_d}{\xi_d})}{\Gamma(1-\frac{\lambda_d}{\xi_d})} e^{\mu_b v} \left((\mu_b+\mu_d)v\right)^{\lambda_b/\xi_d-1} \to \infty,$$

for all $\mu_b > 0$, which leads us to conclude $c_1 = 0$. We deduce c_2 by exploiting the restriction that

$$\int_0^\infty f(v)\, dv = \pi_d, \tag{B.14}$$

where π_d is the probability of positive demand. Hence

$$\pi_d c_2^{-1} = \int_0^\infty e^{-\mu_d v} U\left(1-\tfrac{\lambda_d}{\xi_d}, 2-\tfrac{\lambda_b+\lambda_d}{\xi_d}, (\mu_b+\mu_d)v\right) dv. \tag{B.15}$$

By slightly transforming [25, (3.2.51)], we find

$$c_2^{-1} = \frac{1}{\pi_d} \frac{\Gamma\left(\frac{\lambda_b+\lambda_d}{\xi_d}\right)}{\Gamma\left(1+\frac{\lambda_b}{\xi_d}\right)} {}_2F_1\left(1-\tfrac{\lambda_d}{\xi_d}, 1, 1+\tfrac{\lambda_b}{\xi_d}, -\tfrac{\mu_b}{\mu_d}\right), \tag{B.16}$$

where ${}_2F_1(a_1,a_2,a_3,x) := \sum_{n=0}^\infty \frac{(a_1)_n(a_2)_n}{(a_3)_n n!} x^n$ is the hypergeometric function of Gauss. This proves (4.6).

APPENDIX C: LAPLACE TRANSFORMS FOR COX JUMP SIZES

In this appendix we outline how the differential equation (5.4) is obtained. We take Laplace transforms in (3.2), considering its five terms and calling them T_1, T_2, T_3, T_4 and T_5, successively. Formula (3.2) then translates into

$$T_1 + T_2 + T_3 = T_4 + T_5, \tag{C.1}$$

where

$$T_1 = \lambda_d \int_{v=0}^\infty \mathrm{e}^{-sv} \int_{y=0}^v f(y)\bar{F}_d(v-y)\mathrm{d}y\mathrm{d}v = \lambda_d \phi(s) \frac{1-\mathbb{E}[\mathrm{e}^{-sD}]}{s}, \tag{C.2}$$

$$\begin{aligned} T_2 &= \lambda_d \int_{v=0}^\infty \mathrm{e}^{-sv} \int_{y=0}^\infty g(y)\bar{F}_d(v+y)\mathrm{d}y\mathrm{d}v \\ &= \lambda_d \int_{y=0}^\infty \mathrm{e}^{sy} g(y) \int_{z=y}^\infty \mathrm{e}^{-sz}\bar{F}_d(z)\mathrm{d}z\mathrm{d}y, \end{aligned} \tag{C.3}$$

$$T_3 = \pi_0 \lambda_d \int_0^\infty \mathrm{e}^{-sy}\bar{F}_d(y)\mathrm{d}y, \tag{C.4}$$

$$T_4 = \lambda_b \int_{v=0}^\infty \mathrm{e}^{-sv} \int_{y=v}^\infty f(y)\bar{F}_b(y-v)\mathrm{d}y\mathrm{d}v \tag{C.5}$$

$$= \lambda_b \int_{y=0}^{\infty} \mathrm{e}^{-sy} f(y) \int_{z=0}^{y} \mathrm{e}^{sz} \bar{F}_b(z) \mathrm{d}z \mathrm{d}y,$$

$$T_5 = \xi_d \int_{v=0}^{\infty} v \mathrm{e}^{-sv} f(v) \mathrm{d}v + \alpha_d \phi(s) = -\xi_d \phi'(s) + \alpha_d \phi(s). \tag{C.6}$$

We now evaluate the terms appearing in the righthand sides of (C.2)–(C.5) for the Coxian case of (5.2) and (5.3):

(C.7)

$$\int_{z=0}^{y} \mathrm{e}^{sz} \bar{F}_b(z) \mathrm{d}z = \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1} (1 - p_h) \sum_{j=1}^{i} \prod_{l=1; l \neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} \frac{1}{\beta_j - s} (1 - \mathrm{e}^{(s - \beta_j) y}),$$

(C.8)

$$\int_{z=y}^{\infty} \mathrm{e}^{-sz} \bar{F}_b(z) \mathrm{d}z = \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1} (1 - p_h) \sum_{j=1}^{i} \prod_{l=1; l \neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} \frac{1}{\beta_j + s} \mathrm{e}^{-(s + \beta_j) y},$$

$$\mathbb{E}[\mathrm{e}^{-sB}] = \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1} (1 - p_h) \sum_{j=1}^{i} \prod_{l=1; l \neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} \frac{\beta_j}{\beta_j + s}, \tag{C.9}$$

and hence

$$\frac{1 - \mathbb{E}[\mathrm{e}^{-sB}]}{s} = \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1} (1 - p_h) \sum_{j=1}^{i} \prod_{l=1; l \neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} \frac{1}{\beta_j + s}. \tag{C.10}$$

Combining (C.1) with (C.2)–(C.6), and using (C.7) and the counterparts of (C.8) and (C.10) for $\bar{F}_d(\cdot)$, we find:

$$\begin{aligned} \lambda_d \phi(s) & \sum_{i=1}^{K} q_i \prod_{h=1}^{i-1} (1 - q_h) \sum_{j=1}^{i} \prod_{l=1; l \neq j}^{i} \frac{\delta_l}{\delta_l - \delta_j} \frac{1}{\delta_j + s} \\ & + \lambda_d \sum_{i=1}^{L} q_i \prod_{h=1}^{i-1} (1 - q_h) \sum_{j=1}^{i} \prod_{l=1; l \neq j}^{i} \frac{\delta_l}{\delta_l - \delta_j} \frac{1}{\delta_j + s} [\gamma(\delta_j) + \pi_0] \\ & = \lambda_b \sum_{i=1}^{K} p_i \prod_{h=1}^{i-1} (1 - p_h) \sum_{j=1}^{i} \prod_{l=1; l \neq j}^{i} \frac{\beta_l}{\beta_l - \beta_j} \frac{1}{\beta_j - s} (\phi(s) - \phi(\beta_j)) \\ & - \xi_d \phi'(s) + \alpha_d \phi(s), \end{aligned} \tag{C.11}$$

which is readily rewritten into (5.4).

REMARK 7. *If $\xi_d = 0$, then $\phi(s)$ is obtained from (C.11) in a standard manner; see also Section 6.*

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