

The geometry of Brownian surfaces*

Rémi Léandre

Institut de Mathématiques, Université de Bourgogne, 21000, Dijon, France
e-mail: remi.leandre@u-bourgogne.fr

Abstract: Motivated by Segal’s axiom of conformal field theory, we do a survey on geometrical random fields. We do a history of continuous random fields in order to arrive at a field theoretical analog of Klauder’s quantization in Hamiltonian quantum mechanic by using infinite dimensional Airault-Malliavin Brownian motion.

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1. Introduction

Let us consider m smooth vector fields on R^d , X_i . Suppose that we are in an uniform elliptic situation, that is the following quadratic form

$$\xi \rightarrow \sum \langle X_i(x), \xi \rangle^2 \quad (1)$$

is uniformly invertible on R^d . In such a case, we can introduce the inverse quadratic form $g(x)$ and a measure $dg = \det(g(x))^{-1/2} dx$. These data are invariant under change of coordinates on R^d . A 1-form can be assimilated via the Riemannian metric $g(x)$ to a vector field. We have the following integration by parts formula for any vector field on R^d with compact support:

$$\int_{R^d} X f dg = \int_{R^d} f \operatorname{div} X dg \quad (2)$$

Let $\operatorname{grad} f$ the vector field associated to f via the Riemannian metric. The Laplace Beltrami operator is

$$\Delta = \operatorname{div} \operatorname{grad} \quad (3)$$

Since these data are compatible with change of coordinates, we can consider via local charts a manifold M endowed with a Riemannian structure. The notion of Riemannian measure is intrinsic and the Laplace-Beltrami operator Δ is intrinsically associated to the Riemannian manifold M .

Let us consider m independent Brownian motions B_i on R and the Stratonovitch differential equation:

$$dx_t(x) = \sum X_i(x_t(x)) dB_t^i; \quad x_0(x) = x \quad (4)$$

*This is an original survey paper

It generates a Markov semi-group whose generator is $1/2 \sum X_i^2$. Moreover, there exists a vector field such that:

$$-\sum X_i^2 = \Delta + X_0 \tag{5}$$

Since the Itô formula in Stratonovitch sense is the classical one, (4) is independent of the system of coordinates chosen, and we can consider (4) on a manifold.

For that, let us recall quickly the theory of stochastic processes on a manifold. Let M be a compact manifold. Let $t \rightarrow x_t$ be a random continuous process on the manifold adapted to a filtration F_t endowed with a Probability measure P . $t \rightarrow x_t$ is said to be a semimartingale with values in M if for all smooth function f on M , $t \rightarrow f(x_t)$ is a real semimartingale. Let us consider some smooth vector fields X_i on M . Alternatively, they can be considered as first order differential operators on the space of smooth functions on M satisfying the Leibniz rule. Or they can be considered as smooth of the tangent bundle on M . The solution of the Stratonovitch differential equation

$$dx_t(x) = \sum_{i>0} X_i(x_t(x))dB_t^i + X_0(x_t(x))dt; \quad x_0(x) = x \tag{6}$$

is characterized by the following condition:

$$df(x_t(x)) = \sum_{i>0} X_i f(x_t(x))dB_t^i + X_0 f(x_t(x))dt \tag{7}$$

for each smooth function f , where the vector fields X_i are considered as differential operators. Moreover, if the manifold is compact, the solution of the Stratonovitch differential equation has a unique solution which is a continuous semi-martingale.

If $t \rightarrow x_t$ is a continuous semi-martingale and ω is a smooth 1-form, that is a smooth section of the cotangent bundle of M , we can define the stochastic Stratonovitch integral $\int_0^1 \langle \omega(x_t), dx_t \rangle$ as limit in L^2 of the classical random integral

$$\int_0^1 \langle \omega(x_t^n), dx_t^n \rangle \tag{8}$$

where $t \rightarrow x_t^n$ is a suitable polygonal approximation of $t \rightarrow x_t$. If f is a smooth real function on M , df is a 1-form and the Itô-Stratonovitch formula says that almost surely:

$$f(x_t) - f(x_s) = \int_s^t \langle df(x_u), dx_u \rangle \tag{9}$$

for $s < t$.

Solution of Stratonovitch differential equations realize measure on the *continuous* path space $P(M)$ constituted of continuous paths from $[0, 1]$ into M . This realizes random paths on a manifold. There are two geometries involved:

- The source geometry is the segment $[0, 1]$ (or the circle, if we conditionate by $x_1(x) = x$, that is if we consider the Bridge of the diffusion $t \rightarrow x_t(x)$ or a tree if we consider branching processes).

- The target geometry constituted of the Riemannian manifold M .

If we imbed isometrically the compact manifold M into R^m (It is possible by Nash theorem), we can consider the orthogonal projection $\Pi(x)$ from R^m into $T_x(M)$, the tangent space of M at x . We extend Π for $x \in R^m$. We can consider the diffusion:

$$dx_t(x) = \Pi(x_t(x))dB_t; \quad x_0(x) = x \tag{10}$$

where x belongs to M and B_t is a Brownian motion on R^m . Since the Itô formula in Stratonovitch Calculus is the traditional one, $t \rightarrow x_t(x)$ is a process on M which realizes the Brownian motion on M . It is the Markov process whose generator is the Laplace-Beltrami operator on M . The source geometry is very simple:

- It is $[0, 1]$ if we consider the diffusion. $[0, 1]$ is endowed with its canonical Riemannian structure, but we could choose another Riemannian structure by looking at the equation $dx_t(x) = h(t)\Pi(x_t(x))dB_t$.
- It is the circle if we consider the Brownian bridge.

In theoretical physics, people look at random fields where the source is a Riemannian manifold Σ with possible boundaries and whose target spaces are manifolds (see [46, 47, 48] for surveys). There are two geometries involved in two dimensional field theories:

- The geometry of the world sheet, a Riemannian surface with boundary.
- The geometry of the target manifold.

If we consider the case where the world sheet is a square $[0, 1] \times [0, 1]$ or a cylinder $[0, 1] \times S^1$, the random field $(t, s) \rightarrow x_{t,s}$ realizes an infinite dimensional process $t \rightarrow (s \rightarrow x_{t,s})$ with values in the path space of the manifold or in the loop space of the manifold. We say that we are in presence of a 1+1 dimensional field theory, the first 1 describing the time of the dynamic and the second 1 denoting the dimension of the internal time of the state space. If we consider a more complicated Riemannian surface, the loops are interacting: we refer to the survey of Mandelstam [102] or Witten [126] about that.

There are two possible theories:

- We consider open strings, that is possibly interacting processes on the path space on a manifold.
- We consider closed strings, that is possibly interacting processes on the loop space.

If we consider processes on the loop space, the axioms of Segal [119] of conformal field theory are the followings:

Segal's axiom: Consider the set of possibly disconnected Riemannian surfaces Σ with oriented parametrization $p_i, i \in I$ of the boundary loops, negative (positive) for $i \in I_-(I_+)$ and with a Riemannian structure g , trivial around the boundary. Let us consider an Hilbert space H with an antiunitary involution P .

A real conformal field theory is an assignment

$$(\Sigma, p_i, g) \rightarrow A(\Sigma, p_i, g) \tag{11}$$

where

$$A(\Sigma, p_i, g) : \otimes_{i \in I_-} H \rightarrow \otimes_{i \in I_+} H \quad (12)$$

are trace-classes operators (empty tensor are equal to C) satisfying to the following properties:

Property 1: If (Σ, p_i, g) is the disjoint union of $(\Sigma^\alpha, p_{i_\alpha}^\alpha, g^\alpha)$, then:

$$A(\Sigma, p_i, g) = \otimes_\alpha A(\Sigma^\alpha, p_{i_\alpha}^\alpha, g^\alpha) \quad (13)$$

Property 2: If we reverse the sense of the time of p_{i_0} , we get another loop called \tilde{p}_{i_0} such that if $i_0 \in I_-$:

$$\langle A(\Sigma, \tilde{p}_{i_0}, p_i, g) x_{i_0} \otimes x, y \rangle = \langle A(\Sigma, p_i, g) x, P x_{i_0} \otimes y \rangle \quad (14)$$

Property 3: If F is a conformal diffeomorphism from Σ^1 into Σ^2 , then:

$$A(\Sigma^1, p_i^1, F^* g^2) = A(\Sigma^2, F \circ p_i^1, g^2) \quad (15)$$

Property 4: If Σ' is constructed from Σ by identifying the boundary loops $i_1 \in I_-$ and $i_2 \in I_+$, we have:

$$A(\Sigma', p_{i'}, g) = \text{Tr}_{i_1, i_2} A(\Sigma, p_i, g) \quad (16)$$

where i' is different from i_1 and i_2 and Tr_{i_1, i_2} is the trace between factors i_1 and i_2 in the tensor products of H .

Property 5: If $\tilde{\Sigma}$ is the complex conjugate of Σ , then:

$$A(\tilde{\Sigma}, p_i, g) = A(\Sigma, p_i, g)^* \quad (17)$$

Property 6: If σ is a real smooth function on Σ vanishing in a boundary of the Riemann surface Σ , then:

$$A(\Sigma, p_i, \exp[\sigma]g) = \exp\left[\frac{ci}{2\pi} \int_\Sigma (1/2 \partial\sigma \wedge \bar{\partial}\sigma + R_g \sigma)\right] A(\Sigma, p_i, g) \quad (18)$$

where R_g is the curvature form of the metric g on the surface Σ and c is a constant.

Property 6 is called the conformal anomaly. Namely, these axioms as it was noticed by Gawedzki (Ref. [46], pp. 106–107) may be deduced from the physicist intuitive representative of the amplitude A by formal functional integrals. H is a space of function over the loop space $L(M)$ of the target space M . P arises on the time reversal on the loop $L(M)$ combined with the complex conjugation. The amplitudes A are represented as formal integrals on maps $x : \Sigma \rightarrow M$ fixed on the boundary of Σ (This means we consider a kind of infinite dimensional Brownian bridge):

$$\int_{x \circ p_i = f_i} d\mu(x) = \int_{x \circ p_i = f_i} \exp[-I(x)] dD(x) = A(\Sigma, p_i, g)(f_i) \quad (19)$$

where $dD(x)$ is the formal Lebesgue measure over the set of maps x , and $I(x)$ the energy of x :

$$I(x) = \int_{\Sigma} \langle Dx_S, Dx_S \rangle_{x_S} dg(S) \quad (20)$$

where S denotes the generic element of the Riemannian surface Σ . The quantities (19) lead to infinities, which are regularized by ad-hoc procedures in physics. Let us consider the case where $M = R$ endowed with the constant metric, and in order to simplify, the case where we have no boundary in the Riemannian surface Σ . Let us consider the partition function of the theory:

$$Z = \int_x d\mu(x) \quad (21)$$

and the Gaussian measure $Z^{-1}d\mu(x)$. The random field x is Gaussian and is called the free field. Let us recall some basic backgrounds on Gaussian random fields parametrized by Σ . We suppose in order to simplify that the Riemann surface Σ has no boundary. Let Δ_{Σ} be the Laplacian on Σ . Let be the action

$$I(x(\cdot)) = \int_{\Sigma} \langle (\Delta_{\Sigma} + I)x(\cdot), x(\cdot) \rangle dg \quad (22)$$

Let $x_n(s)$ be the normalized eigenfunctions associated to the eigenvalues λ_n of $\Delta_{\Sigma} + I$. Let us remark that $\lambda_n > 0$ for all n due to I . The Gaussian field associated to $I(x(\cdot))$ can be represented as

$$x(\cdot) = \sum \frac{1}{\lambda_n} B_n x_n(\cdot) \quad (23)$$

where B_n are independent centered normalized Gaussian variables. There are some problems to know the space where the previous series converges. If we consider a smooth function f on M , it can be represented by Sobolev imbedding theorem as

$$f(\cdot) = \sum_n \alpha_n x_n(\cdot) \quad (24)$$

where the sequence α_n is quickly decreasing. This shows us that

$$\int f(S)x(S)dg(S) = \sum \frac{\alpha_n}{\lambda_n} B_n \quad (25)$$

is a convergent series of random variables. This shows us that the random field can be defined as a random distribution.

In order to show that the formal random field defined by (22) is not a true random field, we can compute formally by using (23) $E[x(S)x(S')]$. It is given by

$$\sum \frac{1}{\lambda_n} x_n(S)x_n(S') \quad (26)$$

We recognize in the previous expression the kernel of $(\Delta_{\Sigma} + I)^{-1}$ (or the Green kernel associated to $\Delta_{\Sigma} + I$), which has a logarithmic singularity when $S \rightarrow S'$ $\log d(S, S')$ where d is the Riemannian distance on Σ .

In our situation, we have no mass term I and the treatment is a little bit more complicated due to the presence of zero modes. Let us consider the Laplace-Beltrami operator on M and the associated Green kernel $G(S, S')$. We have $E[x_S x_{S'}] = G(S, S')$. We still have $G(S, S) = \infty$. This explain, following Nelson [108], that the random field x is a generalized process: it is a random distribution [108] and we consider smeared random fields. It is difficult to state what are random distributions with values in a manifold.

But there are simpler random fields than the free field: for instance, the Brownian sheet is a true random field. The goal of this survey is to recall them and to describe the history in order to arrive to a theory of (continuous!) random fields parametrized by any Riemannian surface with values in any target space.

For surveys about the physicists models, we refer to the survey of Witten [126] and Gawedzki [46, 47, 48]. For the relation between analysis on loop space and mathematical physics, we refer to the surveys of Albeverio [3] and Léandre [80, 83].

2. Cairoli equation and Brownian sheet

In this part, we are concerned by a world-sheet which is a square $[0, 1] \times [0, 1]$. Namely, in this case, there is an order on the world-sheet such that we can apply martingale theory in order to study stochastic differential equations.

Let $S = (t, s) \in [0, 1]^2$. We consider the white noise $\eta(S)$. It is a generalized Gaussian process with average 0 and formal covariance given by

$$E[\eta(S)\eta(S')] = \delta_S(S') \quad (27)$$

where $\delta_S(S')$ is the Dirac mass in S . We can consider a measurable set O of the square and we can define

$$B(O) = \int_O \eta(S) dS \quad (28)$$

$B(O)$ is a Gaussian variable of average 0. Moreover,

$$E[B(O)B(O')] = m(O \cap O') \quad (29)$$

where m is the Lebesgue measure on the square.

On the square, there is a natural order. If $S = (t, s)$, we denote by $[0, S] = [0, t] \times [0, s]$ and we can introduce the Brownian sheet:

$$B(S) = \int_{[0, S]} \eta(S') dS' \quad (30)$$

It is a continuous process. Namely, we can use (29) and the fact that $B(S) - B(S')$ is a Gaussian random variable in order to find a positive α such that:

$$E[|B(S) - B(S')|^p]^{1/p} \leq Cd(S, S')^\alpha \quad (31)$$

The result arises by Kolmogorov lemma.

Let $F(S)$ be the σ -algebra spanned by $B(S')$ for $S' \leq S$. If $S_1 > S$, we get almost surely

$$B(S) = E[B(S_1)|F(S)] \quad (32)$$

We say that $s \rightarrow B(S)$ is a martingale with respect to the filtration $F(S)$.

Let $S \rightarrow h(S)$ be a continuous bounded process such that $h(S)$ is $F(S)$ -measurable. We can define the Itô stochastic integral

$$V(S) = \int_{[0,S]} h(S') \delta B(S') \quad (33)$$

$S \rightarrow V(S)$ is still a martingale and Itô's isometry formula states that:

$$E[V(S)^2] = E\left[\int_{[0,S]} h^2(S') dS'\right] \quad (34)$$

We refer to [30] for a theory of Itô integral for multiparameter random processes.

This equality, by using a generalization in the two parameter context of the Peano approximation of a diffusion, allows to consider Cairoli equation [22]: let be $m + 1$ vector fields X_i on R^d with bounded derivatives of all orders and m independent Brownian sheets $B^i(S)$. We consider the two-parameter Itô equation:

$$\delta x(S) = \sum X_i(x(S)) \delta B^i(S) + X_0(S) dS; \quad x(0) = x \quad (35)$$

This means that:

$$x(S) = x + \int_{[0,S]} X_i(x(S')) \delta B^i(S') + \int_{[0,S]} X_0(x(S')) dS' \quad (36)$$

Theorem 1 (Cairoli) *Equation (35) has a unique solution.*

Let us introduce the Hilbert space H_1 of maps from the square into R^m h such that $h(S) = \int_{[0,S]} k(S') dS'$ endowed with the Hilbert structure $\|h\|^2 = \int_{[0,1]^2} |k(S')|^2 dS'$. We remark that:

$$\|h\|^2 = \int_{[0,1]^2} \left| \frac{\partial^2}{\partial s \partial t} h(S) \right|^2 ds \quad (37)$$

(compare with (22)).

Formally, the Brownian sheet is the Gaussian measure on H_1 defined, in a heuristic physicist way by:

$$Z^{-1} \exp[-\|h\|^2/2] dD(h) \quad (38)$$

where $dD(h)$ is the heuristic Lebesgue measure on H_1 . In fact, unlike the free field, which is a generalized process, this measure lives on the space of continuous functions from the square with values in R^m endowed with the uniform norm $\|\cdot\|_\infty$. The link between the physicist heuristic point of view and the rigorous

probabilistic point of view is expressed by large deviations results [11]. Let us consider $\epsilon B(\cdot)$ when $\epsilon \rightarrow 0$. It is given by the formal measure

$$Z_\epsilon^{-1} \exp\left[-\frac{\|h\|^2}{2\epsilon^2}\right] dD(h) \quad (39)$$

Let A be a Borelian subset on the space of continuous functions from the square into R^m , let $\text{Int}A$ be its interior for the uniform topology and $\text{Clos}A$ its closure for the uniform topology. (39) explains heuristically that:

$$-\inf_{h \in \text{Int}A} \|h\|^2 \leq \underline{\lim}_{\epsilon \rightarrow 0} 2\epsilon^2 \log P\{\epsilon B(\cdot) \in A\} \quad (40)$$

$$\overline{\lim}_{\epsilon \rightarrow 0} 2\epsilon^2 \log P\{\epsilon B(\cdot) \in A\} \leq -\inf_{h \in \text{Clos}A} \|h\|^2 \quad (41)$$

Let us consider the equation

$$dx(S)(\epsilon) = \epsilon \sum X_i(x_S(\epsilon)) \delta B^i(S); \quad x(0)(\epsilon) = x \quad (42)$$

and its skeleton:

$$dx(S)(h) = \sum X_i(x(S)(h)) k(S)^i dS; \quad x(0)(h) = x \quad (43)$$

If $\|h\|$ is bounded, the set of solution of (43) is relatively compact in the set of continuous functions from the square into R^d endowed with the uniform topology. This is due to Ascoli theorem.

Doss and Dozzi [29] have shown the following quasi-continuity lemma:

Lemma 2 *For all $a, R, \rho > 0$, there exists α, ϵ_0 such that:*

$$P\{\|x(\cdot)(\epsilon) - x(\cdot)(h)\|_\infty > \rho; \|\epsilon B(\cdot) - h(\cdot)\|_\infty < \alpha\} \leq \exp[-R/\epsilon^2] \quad (44)$$

for $\epsilon \in]0, \epsilon_0]$ and h such that $\|h\|^2 \leq a$.

The proof of this lemma is based upon a generalization in the multiparameter context of the exponential inequality for martingales and of the Cameron-Martin formula.

If A is a Borelian of the set of continuous functions from the square into, R^d , we put $\Lambda(A) = \inf_{x(\cdot)(h) \in A} \|h\|^2$.

Doss-Dozzi [29] deduced the following large deviation estimates for the two-parameter diffusion $x(\cdot)(\epsilon)$.

Theorem 3 (*Wentzel-Freidlin estimates*): *When $\epsilon \rightarrow 0$:*

$$-\Lambda(\text{Int}A) \leq \underline{\lim} 2\epsilon^2 \log P\{x(\cdot)(\epsilon) \in A\} \quad (45)$$

$$\overline{\lim} 2\epsilon^2 \log P\{x(\cdot)(\epsilon) \in A\} \leq -\Lambda(\text{Clos}A) \quad (46)$$

This shows that in order to estimate $P\{x(\cdot)(\epsilon) \in A\}$, we can replace formally $\epsilon B(\cdot)$ by h with the formal measure (38).

We refer to [16, 114] for improvements of this theorem.

Let us suppose that

$$C'|\xi|^2 \geq \sum_{i>0} \langle X_i(x), \xi \rangle^2 \geq C|\xi|^2 \quad (47)$$

for some $C' > C > 0$.

Under this ellipticity hypothesis, Nualart and Sanz [113] have shown by using Malliavin Calculus:

Theorem 4 *The law of $x(S)(\epsilon)$ has a smooth density $q_\epsilon(S)(x, y)$ with respect of the Lebesgue measure on R^d provided that $st \neq 0$.*

Let us put

$$d_S^2(x, y) = \inf_{x(S)(h)=y} \|h\|^2 \quad (48)$$

Under (47), $d_S^2(x, y)$ is finite and continuous in x and y . Léandre and Russo [99] have mixed Malliavin Calculus and Wentzel-Freidlin estimates in order to show:

Theorem 5 (*Varadhan estimates*): *We have if $st \neq 0$ when $\epsilon \rightarrow 0$:*

$$\lim 2\epsilon^2 \log q_\epsilon(S)(x, y) = -d_S^2(x, y) \quad (49)$$

The problem of this theory is that we consider Itô equation. This leads to some problems if we want to constrain the two parameter diffusion to live over a manifold. Norris [110] extending a previous work of Hajek [52] has defined a two-parameter Stratonovitch Calculus which allows him to solve this problem. Namely, Itô formula in two parameters Itô Calculus has no geometrical meaning. Stratonovitch Calculus has a geometrical meaning. This allows Eells-Elworthy-Malliavin to get an intrinsic construction of the Brownian motion on a compact manifold. Norris equation is motivated by a multiparameter extension of the construction of Eells-Elworthy-Malliavin of the Brownian motion on a manifold. This requires the introduction of some geometrical definitions.

Let M be a Riemannian manifold. It inherits the Levi-Civita connection ∇ . It is characterized by the fact that it is a metric connection without torsion. If X is a vector field and Y a vector fields, $\nabla_Y X$ is still a vector field. Moreover,

$$Y \langle X^1, X^2 \rangle = \langle \nabla_Y X^1, X^2 \rangle + \langle X^1, \nabla_Y X^2 \rangle \quad (50)$$

(The Levi-Civita connection preserves the metric) and

$$\nabla_Y X - \nabla_X Y = [X, Y] \quad (51)$$

(The Levi-Civita connection is without torsion). $\nabla_Y X$ is tensorial in Y and a first order operator in X . This means, if f is a smooth function on M

$$\nabla_Y (fX) = Y(f)X + f\nabla_Y X \quad (52)$$

In local coordinates,

$$\nabla_Y X = Y(X) + \Gamma_Y X \quad (53)$$

where Γ denotes the Christoffel symbol matrix.

Is $s \rightarrow x_s$ is a C^1 paths with values in the manifold and $s \rightarrow X_s$ a path in the tangent bundle over x_s , we can define in a intrinsic way:

$$\nabla_s X_s = d/ds X_s + \Gamma_{d/ds x_s} X_s \quad (54)$$

The Levi-Civita connection pass to the cotangent bundle by duality. Moreover, let $O(M)$ be the $O(d)$ principal bundle on M constituted of isometries from R^d into $T_x(M)$. The Levi-Civita connection pass to $O(M)$.

Christoffel symbols and (54) have only a local meaning. In order to perform the computations, we have to see how these quantities behave under local change of coordinates. There is a way to avoid these technicalities. Namely, we can show that all linear bundle E endowed with a linear connection ∇^E is a subbundle of a trivial bundle endowed with the projection connection. Therefore, Christoffel symbols become globally defined.

Eells-Elworthy-Malliavin equation on a Riemannian manifold is

$$dx_t = \tau_t dB_t \quad (55)$$

where B_t is a Brownian motion in the tangent space of M at x and where τ_t is the parallel transport for the Levi-Civita connection. Parallel transport after performing the previous trivialization is given by

$$d\tau_t = -\Gamma_{dx_t} \tau_t \quad (56)$$

τ_t realizes a isometry from the tangent space at x to the tangent space at x_t .

We can generalize this notion: let $s \rightarrow y_s$ be a continuous semi-martingale in M and E a linear bundle on M endowed with a connection ∇^E . The parallel transport along the random path y_t is given by the equation:

$$d\tau_t^E = -\Gamma_{dy_t}^E \tau_t^E \quad (57)$$

If we consider a path $X_t = \tau_t^E h_t$ over y_t , we have

$$\nabla_t^E \tau_t^E h_t = \tau_t^E d/dt h_t \quad (58)$$

where $t \rightarrow H_t$ is a finite energy path in the fiber of E over the starting point of the leading semi-martingale.

After trivializing globally the tangent bundle for the Levi-Civita connection by adding a suitable auxiliary bundle, Eells-Elworthy-Malliavin equation becomes more tractable: it is

$$dx_t = \tau_t dB_t; \quad d\tau_t = -\Gamma_{\tau_t dB_t} \tau_t \quad (59)$$

Let us consider a semi-martingale $V(S)$ defined by (33). $s \rightarrow V(t, s)$ is a semi-martingale and $t \rightarrow V(t, s)$ is a semi-martingale. We can define the Stratonovitch

differential $d_s V(S)$ and $d_t V(S)$ of these semi-martingales. Norris ([110], p. 292) defines too the Stratonovitch differential $d_t d_s V(S)$. These Stratonovitch differentials satisfy to:

$$d_s f(V(S)) = f'(V(S)) d_s V(S) \quad (60)$$

if f is a smooth function on R^d .

$$d_t(V^1(S)d_s V(S)) = d_t V^1(S)d_s V(S) + V^1(S)d_t d_s V(S) \quad (61)$$

for two reals semi-martingales $V^1(S)$ and $V^2(S)$.

$$d_t(d_s V^1(S)d_s V^2(S)) = d_t d_s V^1(S)d_s V^2(S) + d_s V^1(S)d_t d_s V^2(S) \quad (62)$$

(For the various right-brackets for two-parameters semi-martingales involved in (62), we refer to [110].)

But, if we want that the semi-martingale lives on the manifold, we have to look at a more general class of semi-martingales than (33). Let us suppose that:

$$\begin{aligned} V(S) &= x + \int_{[0,S]} h(S') \delta B(S') + \int_{[0,S]} h^1(S') dS' \\ &\quad + \int_{[0,S]} h^2(S') \delta_s B(S') \delta_t B(S') + \int_{[0,S]} h^3(S') d_s \delta_t B(S') \\ &\quad + \int_{[0,S]} h^4(S') dt \delta_s B(S') \end{aligned} \quad (63)$$

We consider Itô integrals in (63). Let us stress the difference between the stochastic integral $\int_{[0,S]} h(S') \delta B(S')$ and $\int_{[0,S]} h(S') \delta_s B(S') \delta_t B(S')$ which is quadratic in the leading Brownian sheet. (60), (61) and (62) remains formally valid for this class of semi-martingales. Let us suppose that $V(S)$ takes its values in M .

Since the Stratonovitch Calculus is the same than the traditional one, by Malliavin's transfer principle, $d_t V(S)$ can be seen formally as a process over $V(S)$ in the tangent bundle. So we cannot define intrinsically $d_s d_t V(S)$. We have to follow the requirement (33) in order to define $\nabla_s d_t V(S)$.

If ω is 1-form on the manifold, we get by Malliavin's transfer principle (see Section 7 about this principle):

$$d_s \omega(d_t V(S)) = \nabla \omega(d_s V(S), d_t V(S)) + \omega(\nabla_s d_t V(S)) \quad (64)$$

Let us recall in order to understand (64) that $\nabla \omega$ is defined by the formula for two vector fields X and Y

$$\nabla \omega(X, Y) = X.\omega(Y) - \omega(\nabla_X Y) \quad (65)$$

where we take the derivative along the vector field X of the function $\omega(Y)$.

Of course, (64) has to be seen at the level of Stratonovitch differential of two parameter semi-martingales.

Let us recall (see [110], p. 317): there exists a unique connection $\hat{\nabla}$ such that:

$$\nabla_s d_t V(S) = \hat{\nabla}_t d_s V(S) \quad (66)$$

$\hat{\nabla}$ preserves the Riemannian metric. Let $\alpha(x, u, v)$ a bundle morphism from the principal bundle $O(M) \oplus O(M)$ over M into the bundle $T(M) \otimes (R^m)^*$ over M . Let $B(S)$ be a R^m -valued Brownian sheet. The main theorem of Norris ([110], p. 318) is the following:

Theorem 6 *The system of equations*

$$\nabla_s d_t x(S) = \alpha(d_s d_t B(S)) \quad (67)$$

$$\nabla_s u(s, t) = 0 \quad (68)$$

$$\hat{\nabla}_t v(s, t) = 0 \quad (69)$$

has a unique solution if $x(0) = x$, $u(t, 0) = u(0, s) = v(t, 0) = v(0, s) = u$ for a semi-martingale $x(S)$ with values in M and two semi-martingales $u(S)$ and $v(S)$ over $x(S)$ in $O(M)$.

3. Ornstein-Uhlenbeck processes on loop spaces and Dirichlet forms

We are concerned in this part by a world-sheet equal to the cylinder $[0, 1] \times S^1$, that is by diffusion processes on the loop space.

Let us consider the Brownian sheet $(t, s) \rightarrow B(t, s)$. It can be considered as an infinite dimensional Brownian motion $t \rightarrow (s \rightarrow B(t, s))$. Let H_1 be the Hilbert space of maps h from $[0, 1]$ into R endowed with the energy norm $\int_0^1 |d/dsh(s)|^2 ds$. The Brownian sheet can be seen formally as the Brownian motion with values in the Hilbert space H_1 . Let h_i be an orthonormal basis of H_1 . H_1 is densely continuously imbedded in $C([0, 1], R)$ the space of continuous function from $[0, 1]$ into R endowed with the uniform norm. Let F be a Fréchet smooth function on $C([0, 1], R)$.

$$\Delta_G F = \sum F''(h_i, h_i) \quad (70)$$

exists and is finite. It is called the Gross Laplacian. Let us remark that the series in (70) converges because we consider an orthonormal basis of H_1 and because F is Fréchet smooth on $C([0, 1], R)$. The Brownian sheet is a Markov process on the path space governed by the Gross Laplacian.

There is another Laplacian on $C([0, 1], R)$. It is the Ornstein-Uhlenbeck operator. Let us endow $C([0, 1], R)$ with the Wiener measure P . Let h in H_1 . We have the following integration by parts formula:

$$E[\langle dF, h \rangle] = E[F \operatorname{div} h] \quad (71)$$

where $\operatorname{div} h = \int_0^1 d/dsh(s) \delta B(s)$ if F is a cylindrical functional

$$F(B(\cdot)) = F^n(B(s_1), \dots, B(s_n)) \quad (72)$$

and B is a one dimensional Brownian motion and F^n a smooth function from R^n into R with bounded derivatives of all orders. We define

$$\langle dF, h \rangle = \sum \frac{\partial}{\partial x_i} F^n(B(s_1), \dots, B(s_n)) h(s_i) \quad (73)$$

such that dF appears as a random element of the dual of H_1 . For a cylindrical functional

$$dF = \sum \frac{\partial}{\partial x_i} F^n(B(s_1), \dots, B(s_n)) 1_{[0, s_i]}(s) \quad (74)$$

dF is called the H -derivative, which is the basic tool of the Malliavin Calculus [112]. Since H_1 is a Hilbert space, we can assimilate dF to a random element $\text{grad } F$ of H_1 . We put

$$\Delta_{O.H} = \text{div grad} \quad (75)$$

The definition is analog to the definition of the Laplace-Beltrami operator in finite dimension (see (3)), the Lebesgue measure which does not exist in infinite dimension being replaced by the Wiener measure.

These operations have an algebraic counterpart. Let $H_1^{sym, n}$ the n^{th} symmetric tensor product associated to H_1 , endowed with its natural Hilbert norm. Let Λ the symmetric Fock space associated to H_1 . To $\sigma = \sum h_n$ belonging to Λ , we associate the Wiener chaoses:

$$H(\sigma) = \sum \int_{[0,1]^n} h_n(s_1, \dots, s_n) \delta B(s_1) \dots \delta B(s_n) \quad (76)$$

(An element of the n^{th} symmetric tensor product of H_1 can be assimilated namely as a symmetric application from $[0, 1]^n$ into R , of finite L^2 norm. The map Wiener chaoses realize an isometry between the symmetric Fock space and $L^2(P)$. Under this identification,

$$\langle dF, H \rangle = A_h F \quad (77)$$

where A_h is the annihilation operator on the symmetric Fock space associated to h . (71) says nothing else that the adjoint of an annihilation operator on the symmetric Fock space is a creation operator and the Ornstein-Uhlenbeck operator is nothing else than the number operator which counts the length of the considered tensor product in the symmetric Fock space (see [104] for an extensive study).

The Ornstein-Uhlenbeck operator (75) is a symmetric positive self-adjoint densely defined operator on $L^2(P)$. It generates therefore a semi-group called the Ornstein-Uhlenbeck semi-group on the Wiener space. Moreover:

$$E[\Delta_{O,H} F.G] = E[\langle \text{grad } F, \text{grad } G \rangle] \quad (78)$$

The right-hand side is called a Dirichlet form. Dirichlet forms are more tractable than Laplacians. There exists an abstract theory of Dirichlet forms and of process related. We refer to the course of Albeverio at Saint-Flour about that [3].

Let E be a topological space, μ be a σ -finite positive measure on E . Let C be a positive, symmetric, densely defined, closed bilinear form on $L^2(\mu)$. Closed means that if $F_n \rightarrow F$ in $L^2(\mu)$ and is a Cauchy sequence for C , F belongs to the domain of C and $C(F^n - F) \rightarrow 0$.

Definition 7 C is called a Dirichlet form if

$$C(\Phi(F)) \leq C(F) \quad (79)$$

for $\Phi(t) = 0$ if $t < 0$, $\Phi(t) = t$ if $t \in [0, 1]$ and $\Phi(t) = 1$ if $t > 1$.

Remark: In fact, we have to regularize Φ (see [3], p. 36).

In general, Dirichlet forms are defined over a dense set and after, via some integration by parts, we perform the closure. (In (78), we consider cylindrical function, and after we perform the completion in order to get a closed Dirichlet form: it is possible, because we have integration by parts (71).) This procedure is an infinite dimensional generalization of the way to get Sobolev spaces in finite dimension, the Lebesgue measure being replaced by the Wiener measure. Namely the historical example of a Dirichlet form is when we take $E = R^d$ endowed with the Lebesgue measure.

$$C(F) = \int_{R^d} \sum_i \left(\frac{\partial}{\partial x_i} F \right)^2 dx \quad (80)$$

The Dirichlet form is closable on R^d because we have integration by parts. The operator which is associated by the analog of (78) in this situation is nothing else than the Laplacian on R^d .

Definition 8 Let C be a Dirichlet form on $L^2(\mu)$. Let O be an open subset of E . We define the capacity of O as followed:

$$\text{Cap } O = \inf_{F \in D(C)} (C(F) + \|F\|_{L^2(\mu)}^2) \quad (81)$$

where $F \geq 1$ almost surely on O . For any subset A of E , we define

$$\text{Cap } A = \inf_{A \subseteq O} \text{Cap } O \quad (82)$$

for O open subset of E .

A property is said to be satisfied quasi-everywhere if the property is satisfied outside a set of capacity 0.

Definition 9 A Dirichlet form is said quasi-regular if:

- (i) There exists compact set F_k such that $\text{Cap}(F_k^c) \rightarrow 0$.
- (ii) There exists a dense subset for C of continuous functions on E , which separates the points of E .

Remark: The definition is in fact more general (see [3], Definition 24).

The basic result is therefore the following: a quasi-regular Dirichlet form defines outside a set of capacity 0 a process $t \rightarrow X_t(x)$. The process is continuous if the Dirichlet form is local:

$$C(F, G) = 0 \quad (83)$$

if the intersection of the support of F and G is of measure 0. The process is defined up a lifetime.

We will produce an example of quasi-regular local Dirichlet form on the based loop space, due to Driver-Roeckner [33].

Let be the heat semi-group $P_t = \exp[-t\Delta]$ on the compact Riemannian manifold M . It is represented by the Brownian motion and has a heat-kernel:

$$P_t f(x) = \int_M p_t(x, y) f(y) dg(y) \tag{84}$$

where $(x, y) \rightarrow p_t(x, y)$ is smooth strictly positive.

dP^x is the law of the Brownian bridge starting from x and coming back at x . $t \rightarrow \gamma(t)$ is a semi-martingale for P^x where $t \rightarrow \gamma(t)$ is the canonical process on $L_x(M)$, the based loop space of M , that is the space of continuous maps γ from S^1 into M such that $\gamma(1) = x$.

The law P^x is characterized by the following property: let $F^n(\gamma(s_1), \dots, \gamma(s_n))$ $s_1 < s_2 < \dots < s_n < 1$ be a cylindrical functional.

$$E[F^n] = p_1(x, x)^{-1} \int_{M^n} p_{s_1}(x, x_1) p_{s_2-s_1}(x_1, x_2) \dots p_{1-s_n}(x_n, x) F(x_1, \dots, x_n) dg(x_1) \dots dg(x_n) \tag{85}$$

Let ∇ be the Levi-Civita connection on M . We can define the parallel transport τ_t along γ_t for the Levi-Civita connection. In local coordinates, it is the solution of the linear Stratonovitch differential equation:

$$d\tau_t = -\Gamma_{d\gamma_t} \tau_t \tag{86}$$

where Γ denotes the Christoffel symbols of the connection (see [35] for a complete theory of horizontal lifts of semi-martingales).

τ_t realizes an isometry from $T_{\gamma(0)}(M)$ into $T_{\gamma(t)}(M)$. We can describe the space where the transport parallel lives in another way: let e_t be the evaluation map $\gamma \rightarrow \gamma(t)$. It is a map from $L_x(M)$ into M . We introduce the pull-back bundle $e_t^*T(M)$ such that τ_t appears as a section of $(e_0^*T(M))^* \otimes e_t^*T(M)$.

The tangent space of differential geometry of a continuous loop γ is realized by the set of continuous sections over S^1 of the bundle on the circle $\gamma^*T(M)$. But this natural tangent bundle of the loop space does not allow to do analysis.

In order to do analysis, we have to use the tangent bundle of Jones-Léandre [65] given in a preliminary form by Bismut [17]. We consider a section of $\gamma^*T(M)$ of the shape $\tau_t h(t) = X(t)$ where $h(\cdot)$ belongs to H_1 , endowed with the energy Hilbert structure $\int_{s_1} |d/dsh(s)|^2 ds = \|h\|^2 < \infty$, with boundary conditions $h(0) = h(1) = 0$. The tangent space of a loop is therefore an Hilbert space. If $h(\cdot)$ is deterministic, we get Bismut's type integration by parts formula [31, 17, 56]

$$E[\langle dF, X \rangle] = E[F \operatorname{div} X] \tag{87}$$

where

$$\operatorname{div} X = \int_0^1 \langle \nabla_s X(s), \delta\gamma(s) \rangle + 1/2 \int_0^1 \langle S_{X(s)}, \delta\gamma(s) \rangle \tag{88}$$

for a cylindrical functional given by a natural generalization to the manifold case by (72). S in (88) is the Ricci tensor (see (117)) associated to the Levi-Civita connection and $\delta\gamma(s)$ the curved Itô integral on the manifold. It is therefore defined by $\int_0^1 \langle \nabla_s X(s), \delta\gamma(s) \rangle = \int_0^1 \langle d/dsh(s), \delta B(s) \rangle$ where $dB(s) = \tau_s^{-1}d\gamma(s)$.

dF appears as a 1-form. Since the tangent space is a Hilbert space, dF can be assimilated by duality as a measurable section $\text{grad } F$ of the stochastic tangent bundle of $L_x(M)$. This point of view is intrinsic and does not require the study of a differential equation leading to the construction of the Brownian motion on M . The main theorem of Driver-Roeckner is the following:

$$C(F) = E[\langle \text{grad } F, \text{grad } F \rangle] \tag{89}$$

defines a quasi-regular local Dirichlet form on the probability space $(L_x(M), P^x)$.

This means that we have to complete the Dirichlet form elementary defined for cylindrical functionals.

As an application of the abstract theory, we can define outside a set of capacity 0 a process $t \rightarrow X_t(\gamma)(s)$ on $L_x(M)$. Moreover, its lifetime is infinite. It is the Ornstein-Uhlenbeck process on the based loop space.

There are several extensions of the work of Driver-Roeckner.

For instance, Albeverio-Léandre-Roeckner [6] consider the free loop space $L(M)$, that is the set of continuous maps γ from the circle into M . Albeverio-Léandre-Roeckner consider the Bismut-Hoegh-Krohn measure on $L(M)$:

$$d\mu = \frac{p_1(x, x)dg(x) \otimes dP^x}{\int_M p_1(x, x)dg(x)} \tag{90}$$

This measure is characterized as follows, if we neglect the normalizing term $\int_M p_1(x, x)dg(x)$, for a cylindrical functional $F = f_1(\gamma(s_1)) \dots f_n(\gamma(s_n))$ by:

$$E[F] = \text{Tr}[\exp[-s_1\Delta]f_1 \exp[-(s_2 - s_1)\Delta]f_2 \dots f_n \exp[-(1 - s_n)\Delta]] \tag{91}$$

Albeverio-Léandre-Roeckner consider vector fields of the same type than before, but with the boundary conditions $\tau_1 h_1 = h_0$ (and not $h(0) = h(1) = 0$ because we consider the free loop space) with the Hilbert structure

$$\int_{S^1} \|h(s)\|^2 ds + \int_{S^1} \|d/dsh(s)\|^2 ds \tag{92}$$

By using the generalization of Bismut's type integration by parts formulas done by Léandre in [78] and [79], Albeverio-Léandre-Roeckner deduce a quasiregular local Dirichlet forms on the free loop space which is a natural extension of the Dirichlet form of (89). Moreover, over the free loop space, there is a natural circle action $\psi_t: \psi_t(\gamma)(s) = \gamma(s + t)$. Due to the cyclicity of the trace in (91), the Bismut-Hoegh-Krohn measure is invariant under the circle action (see [36] for a kind of reciproque) The Dirichlet form of Albeverio-Léandre-Roeckner is invariant under rotation. [6] deduce the existence outside a set of capacity 0 of a

process, with lifetime infinite, invariant by rotation. It is the Ornstein-Uhlenbeck process on the free loop space.

Léandre [80] applies Dirichlet forms theory in order to define the Ornstein-Uhlenbeck process on the universal cover of the based loop space, when $\Pi^2(M)$ is different from 0.

Léandre-Roan [98] consider a developable orbifold M/G where G is a finite group. They use the description of Dixon-Harvey-Vafa-Witten [28] of the free loop space of the developable orbifold, which allows to avoid the difficulty to describe the singularities of the orbifold, in order to construct the Ornstein-Uhlenbeck process on the free loop space of the orbifold (An orbifold has singularities!).

4. Airault-Malliavin equation and infinite dimensional Brownian motion

We are concerned in this case by a world sheet which is $[0, 1] \times N$ or $[0, 1] \times S^1$ where N is a compact manifold. We get a process $t \rightarrow \{S \rightarrow x(t, S)\}$ and the measure on the set of maps from N into the target manifold M $S \rightarrow x(1, S)$ is called following the terminology of Airault-Malliavin the heat-kernel measure. Namely, Airault-Malliavin [2] produce the theory of the Brownian motion on a loop group, which can be easily extended to the manifold case. Airault-Malliavin mechanism allows to produce random fields with parameter space any compact manifold into any compact manifold, with arbitrary regularity as it was extended by Léandre [82].

Let us recall that the theory of processes on infinite dimensional manifolds has a long history: see works of Kuo [74], Belopolskaya-Daletskii [13] and Daletskii [25]. Arnaudon-Paycha [10] have done too a theory of random processes on Hilbert manifolds.

We follow here the presentation of Léandre [82], which generalizes Airault-Malliavin equation to the case of any compact manifold as parameter space and to any compact manifold M as target space.

Let H_1 be an Hilbert space continuously imbedded in the space of continuous functions from N into R , endowed with the uniform topology. Let S be the generic element of N and h be a generic element of H_1 . We suppose:

Hypothesis H: There exists a map $(S, S') \rightarrow e_S(S')$ such that:

- (i) $h(S) = \langle h, e_S \rangle_{H_1}$
- (ii) $(S, S') \rightarrow e_S(S')$ is Hoelder with Hoelder exponent α .

We consider the Brownian motion $t \rightarrow B_t(\cdot)$ with values in H_1 .

$t \rightarrow B_t(S)$ is a R -valued finite dimensional Brownian motion and the right-bracket between $B_t(S)$ and $B_t(S')$ satisfies to:

$$\langle B_t(S), B_t(S') \rangle = t e_S(S') \quad (93)$$

By Kolmogorov lemma [103], we deduce that $(t, S) \rightarrow B_t(S)$ has a version which is Hoelder.

As a matter of fact, we can consider an orthonormal basis h_i of H_1 such that

$$B_t(S) = \sum B_i(t)h_i(S) \quad (94)$$

where $B_i(\cdot)$ are some independent R -valued Brownian motions. We have

$$e_S(S') = \sum h_i(S)h_i(S') \quad (95)$$

But the series does not converge in H_1 , but in a convenient space of Hoelder functions by **Hypothesis H**. We will describe the situation more precisely later.

We consider Airault-Malliavin equation

$$d_t x_t(x)(S) = \Pi(x_t(x)(S))d_t B_t(S); \quad x_0(x)(S) = x \quad (96)$$

where $B_t(\cdot)$ is a collection of m independent Brownian motion in H_1 still denoted $B_t(\cdot)$.

This realizes a family of Brownian motion $t \rightarrow x_t(x)(S)$ in the manifold M .

By using (93) and the Kolmogorov Lemma [103], we get ([82] Theorem 2.1):

Theorem 10 $S \rightarrow x_1(S)$ has almost surely a version which is $\alpha/2 - \epsilon$ Hoelder.

Scheme of the proof: We have:

$$\begin{aligned} d_t x_t(x)(S) - d_t x_t(x)(S') &= (\Pi(x_t(x)(S)) - \Pi(x_t(x)(S')))d_t B_t(S) \\ &\quad + \Pi(x_t(x)(S'))(d_t B_t(S) - d_t B_t(S')) \end{aligned} \quad (97)$$

From Burkholder-Davies-Gundy inequality and from (93) we deduce that

$$\begin{aligned} E[|x_t(x)(S) - x_t(x)(S')|^p] \\ \leq C \int_0^t E[|x_s(x)(S) - x_s(x)(S')|^p]ds + d(S, S')^{\alpha p/2} \end{aligned} \quad (98)$$

The result arises by Kolmogorov lemma [103] and Gronwall lemma.

Remark: [87] has considered the Sobolev space $H_k(N)$ of functions from N into R

$$\int_N h(S)(\Delta_N + 1)^k h(S) dm(S) = \|h\|_k^2 \quad (99)$$

where m is the Riemannian measure associated to a Riemannian structure on N and Δ_N is the Laplace-Beltrami operator on N . The main property [51] is that C^k norms of functions can be estimated by their Sobolev norms in $H_{p(k)}$ where $p(k)$ tends to infinity when $k \rightarrow \infty$. Let h_n be the sequence of eigenvectors of $\Delta_N + 1$ associated to the eigenvalues λ_n . When $n \rightarrow \infty$, $\lambda_n \sim Cn^\beta$ [51]. The Gaussian field associated to (99) can be written as

$$\sum \frac{1}{\lambda_n^k} B_n h_n = B(\cdot) \quad (100)$$

where B_n are independent centered normalized Gaussian variables. $E[\|h\|_{k'}^2] < \infty$ for some big k' if k is big enough. We apply Sobolev imbedding theorem [51].

Let k_0 be given. If k is big enough, the Brownian motion with values in $H_k(N)$ is C^{k_0+1} [100] and $S \rightarrow x_1(S)$ is almost surely of class C^{k_0} due to the final dimensional Sobolev imbedding theorem, by an argument similar to the proof of Theorem 10.

Theorem 10 produces a generalization of Airault-Malliavin equation on a loop group: $N = S^1$, $M = G$ is a compact Lie group, H_1 is a convenient Sobolev space of maps from S^1 into the Lie algebra of G and

$$d_t g_t(s) = g_t(s) d_t B_t(s) \tag{101}$$

is the equation of the Brownian motion on the group G , when s is fixed [2].

According to the terminology of Airault-Malliavin, the law of $S \rightarrow x_1(S)$ is called the heat-kernel measure on the set of maps from N into M .

We would like to define the family of stochastic differential equations (96) as a process on the set of maps from N into M . This requires a theory of random process on Banach manifolds, and, therefore, a theory of processes on Banach spaces. If we consider the set of maps from N into R endowed with some Hoelder norm, it is a very bad Banach space, because the Hoelder norm is very irregular. The good understanding of Stratonovitch differential equations on Banach spaces needs, following Brzezniak-Elworthy [19] the introduction of M -2 Banach manifolds. See [19] for the general theory.

We consider the space H_1 of maps h from S^1 into R^m endowed with the Hilbert structure:

$$\int_{S^1} |h(s)|^2 ds + \int_{S^1} |d/dsh(s)|^2 ds = \|h\|^2 \tag{102}$$

Let $W^{\theta,p}$ be the sets of maps from S^1 into R^m endowed with the Banach structure:

$$\|h\|_{\theta,p} = \left(\int_{S^1} |h(s)|^p ds + \int_{S^1 \times S^1} \frac{|h(s_1) - h(s_2)|^p}{|s_1 - s_2|^{1+\theta p}} ds_1 ds_2 \right)^{1/p} \tag{103}$$

The interest of the norm $\|\cdot\|_{\theta,p}$ is that it contains only integrals in its definition and not supremum as the Hoelder norm. In this point of view, the Sobolev-Slobodetski space $W^{\theta,p}$ is more interesting than an Hoelder Banach space. Moreover, if $1/p < \theta < 1$, the Brownian motion $B_t(\cdot)$ with values in H_1 takes its values in $W^{\theta,p}$ [19]. The main ingredient of Brzezniak-Elworthy theory is:

Theorem 11 *Let F be the Nemytski map $h(\cdot) \rightarrow (s \rightarrow \Pi(h(s)))$. Then F is Fréchet smooth from $W^{\theta,p}$ into himself and has linear growth:*

$$\|F(h(\cdot))\|_{\theta,p} \leq A + B\|h(\cdot)\|_{\theta,p} \tag{104}$$

The idea of Brzezniak-Elworthy is to consider the family indexed by S^1 of Stratonovitch differential equations on M

$$dx_t(s) = \Pi(x_t(s)) d_t B_t(s) \tag{105}$$

as a unique Stratonovitch differential equation on $W^{\theta,p}$:

$$dX_t(\cdot) = F(X_t(\cdot))d_t B_t(\cdot) \tag{106}$$

If there exists something like a nice martingale theory on $W^{\theta,p}$, we could prove, by reproducing the arguments of the theory of Stratonovitch differential equation on a finite dimensional manifold, by using Theorem 11, the following theorem:

Theorem 12 (106) *defines a Markov process on $W^{\theta,p}$ if $p \in [2, \infty[, \theta \in]0, 1[$ and $1/p < \theta < 1$. Moreover, if the starting loops γ belongs to $L(M)$, the free loop space of M , $t \rightarrow X_t(\gamma)$ belongs to $L(M)$.*

In order to prove this theorem, Brzezniak-Elworthy use the fact that $W^{\theta,p}$ is a $M-2$ Banach space if $p \in [2, \infty[$ and $\theta \in]0, 1[$. Let us recall what is a $M-2$ Banach space E .

A finite process indexed by the positive integers M_k is called a martingale with values in E respectively to the filtration F_k if

$$E[M_{k'}|F_k] = M_k \tag{107}$$

if $k' > k$. We won't describe the natural integrability conditions which appear in the definition of a E -valued martingale (for instance $E[|M_k|^p] < \infty$). But, there is one which is satisfied for only special Banach spaces:

Definition 13 *A Banach space is called $M-2$ if there exists a constant $C_2(E)$ such that for all martingales M_k :*

$$\sup_k E[|M_k|^2] \leq C_2(E) \sum_k E[|M_k - M_{k-1}|^2] \tag{108}$$

Let \tilde{H} be an Hilbert space. We consider the formal Gaussian measure:

$$\frac{1}{Z} \exp(-\|\tilde{h}\|^2/2)dD(\tilde{h}) \tag{109}$$

with the formal Lebesgue measure $dD(\tilde{h})$. We suppose, as we have done several times, that there exists an inclusion i from \tilde{H} into E such that $i(\tilde{H})$ is dense in E such that the Gaussian measure lives on E . We say that we are in presence of an abstract Wiener space [75]. Let us consider the Brownian motion \tilde{B}_t with values in \tilde{H} . It takes in fact its values in E . Let ξ be a continuous linear map from E into E . We deduce $\xi \circ i$ a continuous linear map from \tilde{H} into E .

Let $\xi(t_k)$ be random continuous maps from E into E , F_{t_k} measurable for the σ -algebra spanned by the Brownian motion \tilde{B}_t for $t \leq t_k$. We can define the elementary Itô integral with values in E :

$$I(x) = \sum_{k=0}^{n-1} \xi(t_k)(\tilde{B}_{t_{k+1}} - \tilde{B}_{t_k}) \tag{110}$$

From (108), since $t \rightarrow \tilde{B}_t$ is a martingale with values in E , we get:

$$E[|I(\xi)|^2] \leq C_2(E)E\left[\int_0^1 |\xi(t)|_{L(\tilde{H},E)}^2 dt\right] \quad (111)$$

where $\xi(t) = \xi(t_k)$ if $t \in [t_k, t_{k+1}[$, a convenient subdivision of $[0, 1]$.

Since we have the important property (111), Brzezniak-Elworthy can extend words by words the technics available for finite dimensional stochastic differential equation to the case of differential equations on Banach manifolds modelled on Banach spaces (in the Fréchet sense) of type $M-2$, if they consider **Fréchet-smooth** vector fields. (106) becomes a special case of this general theory.

We can characterize geometrically $M-2$ Banach spaces [115]. Pisier's characterization theorem allows to show that $W^{\theta,p}$ is a $M-2$ Banach space under the previous considerations.

Let us indicate the road of possible developments.

Maier-Neeb ([101]) have defined a central extension by a finite dimensional commutative Lie group of a current group of maps from N into G , a compact Lie group. It is possible to define of this central extension a structure of $M-2$ Banach Lie group modelled on a suitable Besov-Slobodetski space. We consider the Brownian Motion $B_t(\cdot)$ in some Besov-Slobodetski space of maps from N into the Lie algebra of G and the Brownian motion $B^1(t)$ (finite dimensional) in the Lie algebra of the finite dimensional commutative Lie group giving the central extension of the current group. We can apply Brzezniak-Elworthy theory in this situation in order to study the stochastic differential equation on the central extension of the current group:

$$d\hat{g}_t = \hat{g}_t d\hat{B}_t \quad (112)$$

where $\hat{B}_t = B_t(\cdot) \oplus B_t^1$.

Léandre [84] consider a world sheet Σ with a random Riemannian structure. In order to define the measure on the set of metrics g on the world-sheet, Léandre uses Faddeev-Popov procedure, which allows to define the measure on the Teichmüller space, and the Shavgulidze measure on the gauge group [120]. This allows Léandre to produce a stochastic Polyakov measure, with a unitary action of the gauge group (the spaces of diffeomorphism of Σ isotopic to the identity) on the L^2 of the theory (see [5] for a mathematical introduction to physicists bosonic string theory).

In fact, physicists are not only interested by Bosons, but too by Fermions. Let $\Lambda(R^n)$ be the exterior algebra of R^n . Let $\theta_I = \theta_{i_1} \wedge \dots \wedge \theta_{i_I}$ if $I = (i_1, \dots, i_I)$ where θ_i is the canonical basis of R^n . Let $F(\theta) = \sum a_I \theta_I$. The Berezin integral is defined by:

$$\int_B F(\theta) = a_{1,2,\dots,n-1,n} \quad (113)$$

Physicists are interested by Fermionic Gaussian processes. Alvarez-Gaume [7], has deduced a super-symmetric proof of the Index theorem. A. Rogers [117] has defined a super-Brownian motion on a finite dimensional manifold which

allows to give a rigorous version of the formal proof of Alvarez-Gaume of the Index theorem on a finite dimensional manifold. Witten [127] is motivated by the Index theorem on the free loop space of a manifold. He is motivated in fact by super-symmetric two-dimensional field theories. Motivated by [127], Léandre [86] has done an infinite dimensional extension of the work of A. Rogers and has defined the super-Brownian motion on a loop group.

5. Logarithmic Sobolev inequality and heat kernel measure

Let us consider a finite dimensional compact Riemannian manifold M . Let dg be the Riemannian probability measure on M and Δ_M be the Laplace-Beltrami operator on M . It has a discrete spectrum and can be diagonalized [51]. This implies that there is a spectral gap [26]. This means that:

$$\int_M f^2 dg - \left(\int_M f dg\right)^2 \leq C \int_M f \Delta_M f dg \quad (114)$$

for any function f on M for a suitable constant C . Spectral gap inequality is implied by Logarithmic Sobolev inequality [26]: If $\int_M f^2 dg = 1$

$$\int_M f^2 \log f^2 dg \leq C \int_M |\text{grad } f|^2 dg \quad (115)$$

We had seen that Dirichlet forms can be defined in infinite dimension. In particular, it is known since a long time that the Dirichlet form (3.6) satisfies a Logarithmic Sobolev inequality. This implies that the Ornstein-Uhlenbeck operator on the Wiener space has a spectral gap (It is in this case obvious, because we can diagonalize the Ornstein-Uhlenbeck operator, which is too the Number operator on the Symmetric Fock space).

Let dP_x be the law of the Brownian motion starting from x in the compact manifold. $s \rightarrow \gamma_s$ denotes the canonical process on the based path space of paths starting from x and $s \rightarrow \tau_s$ the stochastic parallel transport for the Levi-Civita connection called ∇ .

The curvature tensor is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z = R(X, Y)Z \quad (116)$$

for vector fields X, Y, Z on M . It is tensorial in all its arguments. The Ricci curvature is given by:

$$S(Y) = \text{Tr}R(., Y)(.) = - \sum R(e_i, Y)e_i \quad (117)$$

where e_i is an orthonormal basis of $T_x(M)$.

Bismut [17] considered the following differential equation

$$\tau_s^{-1} \nabla_s X_s^B(h) = d/dsh(s)ds - 1/2\tau_s^{-1} S(X_s^B(h))ds \quad (118)$$

where h belongs to the Cameron-Martin space of maps from $[0, 1]$ into $T_x(M)$ such that $h(0) = 0$ and such that $\int_0^1 |d/dsh(s)|^2 ds < \infty$. (We refer to (54) for the notation $\nabla_s X_s$.) If $X_s^B(h) = \tau_s H_s$, (118) means that $dH_s = dh_s - 1/2\tau_s^{-1}S(X_s^B(h))ds$.

Let us explain the mysterious appearance of the Ricci tensor in (118) or in (88). We would like to compute $\nabla_X \tau_s^{-1} d\gamma_s$ where $X_t = \tau_t h_t$. For that we have to take the derivative of the parallel transport. $e_t^* T(M)$ appears as bundle on the path space, which inherits from a connection ∇^∞ from the Levi-Civita connection on the tangent bundle of M . If we use the considerations following (88), we see that:

$$\nabla_X \tau_t^{-1} d\gamma_t = \nabla_X^\infty \tau_t^{-1} d\gamma_t + \tau_t^{-1} \tau_t dh_t \quad (119)$$

By Bismut-Araf'evas formula [9, 17]

$$\nabla_X^\infty = \tau_t \int_0^t \tau_s^{-1} R(d\gamma_s, X_s) \tau_s \quad (120)$$

such that

$$\nabla_X \tau_t^{-1} d\gamma_t = dh_t - \int_0^t \tau_s^{-1} R(d\gamma_s, X_s) \tau_s dB_t \quad (121)$$

We have used the theory of Eells-Elworthy-Malliavin, which says that

$$d\gamma_s = \tau_s dB_s \quad (122)$$

where dB_s is the Stratonovitch differential of a flat Brownian motion in $T_x(M)$.

The Ricci tensor appears when we convert the last Stratonovitch integral in an Itô integral. Tangent vectors of Jones-Léandre [65] are simple to write, but the Ricci tensor appears in the integration by part formula. For Bismut's tangent vector, vector fields are complicated to write but the integration by parts formula is simple to write. Namely, for a cylindrical functional, Bismut established the integration by parts formula [17]

$$\begin{aligned} E[\langle dF^B, X^B(h) \rangle] &= E[\langle \text{grad}^B F, X^B(h) \rangle] \\ &= E[F^B \int_0^1 \langle \tau_s d/dsh(s), \delta\gamma(s) \rangle] \\ &= E[F^B \int_0^1 \langle d/dsh(s), \delta B_S \rangle] \end{aligned} \quad (123)$$

This allows Fang-Malliavin [38] to establish the following Clark-Ocone formula:

$$F(\gamma) = E[F] = \int_0^1 \langle E[k_s | F_s], \delta B_s \rangle \quad (124)$$

where $\langle dF^B, X^B(b) \rangle = \langle k, h \rangle_{H_1}$ and where F_s is the filtration spanned by the curved Brownian motion. Let us explain in more details the difference

between Bismut's way of differential Calculus on the path space and the way defined in Section 3 (see [79] for an extensive study of that).

We consider the equation

$$d\tau_t^B = -\Gamma_{d\gamma_t}\tau_t^B - 1/2S_{\tau_t^B}dt \quad (125)$$

starting from identity. A Bismut's vector field is written has $X_t^B(h) = \tau_t^B h_t$ with Hilbert metric

$$\|X^B(h)\|^2 = \int_0^1 |d/dsh_s|^2 ds \quad (126)$$

For a cylindrical functional $F = F^n(\gamma_{s_1}, \dots, \gamma_{s_n})$

$$dF^B = \sum \langle \text{grad}_{\gamma_{s_i}} F^n(\gamma_{s_1}, \dots, \gamma_{s_n}), \tau_{s_i}^B 1_{[0, s_i]}(s) \cdot \rangle \quad (127)$$

Clark-Ocone formula allows to Capitaine-Hsu-Ledoux [23] to show, by a simple use of the Itô formula, that, if $E[F^2] = 1$

$$E[F^2 \log F^2] \leq CE[\|\text{grad}^B F\|^2] \quad (128)$$

On the other hand, Léandre [78], p. 521, has remarked that the tangent space of Bismut of the Brownian loop is isomorphic to Jones-Léandre tangent space [65] and that the norm of the isomorphism as well as its inverse is bounded. This allows Capitaine-Hsu-Ledoux [23] to show the following theorem:

Theorem 14 $E[F^2 \log F^2] \leq CE[\|\text{grad} F\|^2]$ if $E[F^2] = 1$ where we follow the notations of (78).

This constitutes a Logarithmic Sobolev inequality on the continuous based path space, whose first proof was done by Hsu [55]. Léandre [81] has done an analogous inequality on C^1 paths, by using Capitaine-Hsu-Ledoux method.

For material about logarithmic Sobolev inequality, we refer to [8].

We are motivated by an infinite dimensional generalization of this proof. We consider the based loop group $L_e(G)$ of continuous maps from S^1 into G starting from the identity.

According [2], the tangent space of a loop g is constituted of loops of the type $X_s(h) = g_s h_s$ where h is a finite energy loop in the Lie algebra of G starting from 0. We take as Hilbert norm

$$\|X.(h)\|^2 = \int_{S^1} |d/dsh_s|^2 ds \quad (129)$$

We can compute easily the bracket of two vector fields and show it is still a vector field. Therefore, by an abstract argument, there is a Levi-Civita connection on $L_e(G)$. We can compute its curvature. Freed [41] has succeeded to construct the Ricci tensor associated to the Levi-Civita connection on the based loop group, because we are in one dimension. These ingredients allow to Fang [37] to repeat the proof of Capitaine-Hsu-Ledoux of logarithmic Sobolev inequalities, but for the Brownian motion on a loop group instead of the Brownian motion on a

finite dimensional compact manifold. Fang consider the Brownian motion $B_t(\cdot)$ with values in H_1 , the space of based loop in the Lie algebra of G given by the Hilbert structure (129) and consider Airault-Malliavin equation:

$$d_t g_t(s) = g_t(s) d_t B_t(s); \quad g_0(s) = e \quad (130)$$

Fang consider the heat-kernel measure on $L_e(G)$ given by the law of $s \rightarrow g_1(s)$. Fang states for the heat-kernel measure the following integration by parts formula valid for cylindrical functionals:

$$E[\langle dF, X(h) \rangle] = E[F \operatorname{div} X(h)] \quad (131)$$

This allows to define a Dirichlet form $E[\langle \operatorname{grad} F^1, \operatorname{grad} F^2 \rangle]$. Fang [37], by using an infinite dimensional generalization of the method of Capitaine-Hsu-Ledoux [23] established the following theorem:

Theorem 15 *For the heat-kernel measure on the based loop-group, we have if $E[F^2] = 1$*

$$E[F^2 \log F^2] \leq CE[\|\operatorname{grad} F\|^2] \quad (132)$$

We refer to [32, 62] and [63] for various statements about Logarithmic-Sobolev inequalities for heat-kernel measure on loop groups.

6. Wentzel-Freidlin Estimates

Let us look at the situation of Theorem 10:

Let \tilde{H} the Hilbert space of maps from $[0, 1]$ into $H_2 = H_1 \oplus \cdots \oplus H_1$ endowed with the Hilbert structure:

$$\int_0^1 \|d/dth_t\|_{H_2}^2 = \|h\|^2 \quad (133)$$

if $h_t \in H_2$.

The Brownian motion $B_t(\cdot)$ with values in H_2 has formally as law the Gaussian measure on \tilde{H}

$$\frac{1}{Z} \exp(-\|h\|^2/2) dD(h) \quad (134)$$

where $dD(h)$ is the formal Lebesgue measure on \tilde{H} . This statement is clarified by the following large deviations estimates. Let $\epsilon B_t(\cdot)$ the Brownian motion driven by a small parameter ϵ . Following (40), (41), let A be a Borelian subset of the set of maps from $[0, 1] \times N$ into R^m endowed with the uniform topology. $\operatorname{Int} A$ denotes its interior for the uniform topology, and $\operatorname{Clos} A$ its closure for the uniform topology. We get, analogously to (40), (41), when $\epsilon \rightarrow 0$:

$$- \inf_{h \in \operatorname{Int} A} \|h\|^2 \leq \underline{\lim} 2\epsilon^2 P\{\epsilon B_t(\cdot) \in A\} \quad (135)$$

$$\overline{\lim} 2\epsilon^2 P\{\epsilon B_t(\cdot) \in A\} \leq - \inf_{h \in \operatorname{Clos} A} \|h\|^2 \quad (136)$$

This statement has an abstract counterpart: let (H, E) be an abstract Wiener space. Let e be the generic element of E and $d\mu$ the Gaussian measure on E associated to this abstract Wiener space. Let $\|\cdot\|$ be the Hilbert norm on H and $|\cdot|$ the Banach norm on E . Let A be a Borelian subset of E , $\text{Int } A$ its interior for the metric $|\cdot|$ and $\text{Clos } A$ its closure for the metric $|\cdot|$. We have the analog of (135) and (136). When $\epsilon \rightarrow 0$

$$- \inf_{h \in \text{Int } A} \|h\|^2 \leq \underline{\lim} 2\epsilon^2 \mu\{\epsilon e \in A\} \quad (137)$$

$$\overline{\lim} 2\epsilon^2 \mu\{\epsilon e \in A\} \leq - \inf_{h \in \text{Clos } A} \|h\|^2 \quad (138)$$

This abstract theorem can be for instance applied to the free field, where E can be chosen as a negative order Sobolev space.

We consider Airault-Malliavin equation submitted to a small parameter ϵ :

$$dx_t(\epsilon)(S) = \epsilon \Pi(x_t(\epsilon)(S)) dB_t(S); \quad x_0(\epsilon)(S) = x \quad (139)$$

We cannot apply directly the abstract previous abstract results, because the solution of (139) is only almost surely defined. But we get the following lemma, analogous to Lemma 2.:

Lemma 16 *For all $a, R, \rho > 0$, there exists α, ϵ_0 such that:*

$$P\{\|x(\epsilon)(\cdot) - x(h)(\cdot)\|_\infty > \rho; \|\epsilon B(\cdot) - h(\cdot)\|_\infty < \alpha\} \leq \exp[-R/\epsilon^2] \quad (140)$$

for $\epsilon \in]0, \epsilon_0]$ and h such that $\|h\|^2 \leq a$ where

$$dx_t(h)(S) = \Pi(x_t(h)(S)) d_t h_t(S); \quad x_0(h) = x \quad (141)$$

Scheme of the proof: It is based upon an unwritten idea of Kusuoka [76] in order to prove large deviation estimates for flows. By a result of [11] (see [77] for a simple proof), we get: For all $a, R, \rho > 0$, there exists α, ϵ_0 such that:

$$P\{\|x(\epsilon)(S) - x(h)(S)\|_\infty > \rho; \|\epsilon B(\cdot) - h(\cdot)\|_\infty < \alpha\} \leq \exp[-R/\epsilon^2] \quad (142)$$

for $\epsilon \in]0, \epsilon_0]$ and h such that $\|h\|^2 \leq a$.

We define over N a small triangulation where there are $\exp[C\epsilon^{-2}]$ sites S_i . We apply (142) to each site. We remark, by Kolmogorov lemma, that for some η :

$$P\left\{ \sup_{t, S, S'} \frac{|x_t(\epsilon)(S) - x_t(\epsilon)(S')|}{d(S, S')^\eta} > R \exp[C'\epsilon^{-2}] \right\} \quad (143)$$

is smaller than $\exp[-K\epsilon^{-2}]$ for any K .

Definition 17 *If A is a Borelian subset of the continuous maps from $[0, 1] \times N$ into M endowed with the uniform topology, we put:*

$$\Lambda(A) = \inf_{x, (h)(\cdot) \in A} \|h\|^2 \quad (144)$$

We get (see [82]):

Theorem 18 (*Wentzel-Freidlin estimates*). *When $\epsilon \rightarrow 0$*

$$-\Lambda(\text{Int } A) \leq \underline{\lim} 2\epsilon^2 \log P\{x(\epsilon)(\cdot) \in A\} \quad (145)$$

$$\overline{\lim} 2\epsilon^2 \log P\{x(\epsilon)(\cdot) \in A\} \leq -\Lambda(\text{Clos } A) \quad (146)$$

For another proof, in the case of the Brownian motion on a loop group, we refer to the work of Fang-Zhang [39].

Moreover, in the last developments of string theory, [27], people look at branes: it is a submanifold M_1 of the target manifold M . If Σ is a Riemann surface with boundary $\partial\Sigma$, people look at maps from Σ into M such that $\partial\Sigma$ is mapped on M_1 . Léandre [88] consider the Brownian motion (in the sense of Airault-Malliavin) on the set of cylinders attached to a brane M_1 and performs the large deviation theory.

7. Stochastic Wess-Zumino-Novikov-Witten model and stochastic field theories

In fact, Felder-Gawdzki-Kupiainen [40] have shown that the Hilbert space H of Segal's axioms should be the Hilbert space on L^2 -sections of a convenient line bundle on the loop space, endowed with a convenient formal measure, and that the amplitudes of the theory should be connected with a generalized parallel transport of elements of this line bundle along the surface, which realizes the (interacting!) dynamics of the loop space.

Line bundles over loop spaces can be seen by two ways:

- Either the loop space is simply connected (this means that $\Pi^1(M) = \Pi^2(M) = 0$) and the line bundle is determined by its curvature.
- Either the loop space is not simply connected. There are constructions of line bundle of the loop space associated to Deligne cohomology of the manifold [45], gerbes [18] and bundle gerbes [49].

The line bundle on the loop space has to satisfy the so-called fusion property: let be two based loop in x γ^1 and γ^2 . We can construct, since the loop are based on the same point, a big loop γ . There is a map $\pi^1 : \gamma \rightarrow \gamma^1$ and $\pi^2 : \gamma \rightarrow \gamma^2$. We require that:

$$\xi(\gamma) = \pi^{1*}\xi(\gamma^1) \otimes \pi^{2*}\xi(\gamma^2) \quad (147)$$

Moreover, there are basically two theories:

- One on a compact Lie group. It is called the Wess-Zumino-Novikov-Witten model.
- One on a general compact manifold.

This part is relevant to the so-called **Malliavin's transfer principle**: A formula which is true in the deterministic context and which has a meaning via Stratonovitch stochastic Calculus, is still valid, but almost surely.

The philosophy of this part is to replace the kinetic term of conformal field theory by the more tractable one arising from Airault-Malliavin equation. The geometry is on the other hand very similar to the geometry developed in the surveys of Gawedzki [46, 47, 48]. Moreover, we can consider 1+2 dimensional theory, and the heat kernel measure associated or 1+1 dimensional theory.

We consider the torus $T^2 = S^1 \times S^1$. We consider the Hilbert space H_1 of maps $h(\cdot)$ from T^2 into the Lie algebra of a compact Lie group

$$\|h\|^2 = \int_{T^2} \langle h(S), (-\frac{\partial^2}{\partial s^2} + 1)(-\frac{\partial^2}{\partial t^2} + 1)h(S) \rangle dS \quad (148)$$

if $S = (s, t)$.

This Hilbert satisfies to hypothesis H of Section 4. The Green kernel $E_S(S')$ is in this situation the product of the 1-dimensional Green kernels associated to the operator on the circle $-\frac{\partial^2}{\partial s^2} + 1$ which are equal to

$$e_0(s) = \lambda \exp[-s] + \mu \exp[s] \quad (149)$$

for $0 \leq s \leq 1$ such that $e(0) = e(1)$. $e_s(s')$ is got by translation (see [85], p. 5534). Therefore $E_S(S')$ satisfies Hypothesis H of Section 4.

We consider Airault-Malliavin equation:

$$d_u g_u(S) = g_u(S) d_u B_u(S); \quad g_0(S) = e \quad (150)$$

where $u \rightarrow B_u(\cdot)$ is a Brownian motion with values in H_1 . $S \rightarrow g_1(S)$ defines the heat-kernel measure $d\mu$ on the Hoelder torus group of Hoelder maps from T^2 into G [85]. But, as a matter, of fact, $d\mu$ defines a measure on the strong Hoelder torus group $T_{\epsilon, *}(G)$ of maps such that:

$$\lim_{S \rightarrow S'} \frac{d(g(S), g(S'))}{d(S, S')^\epsilon} = 0 \quad (151)$$

It is an infinite dimensional manifold. By the general theory of Bonic-Frampton-Tromba [14, 15], there are Fréchet smooth partition of unity associated to a cover of the strong Hoelder torus group by open subsets V_i which is locally finite.

We get a generalization of Theorem 11 in this context [85]:

Theorem 19 *Let f be a map from $T^2 \times G$ into G (conveniently imbedded into R^m) with bounded derivatives of all orders. Let F be the Nemytski map:*

$$g(\cdot) \rightarrow (S \rightarrow f(S, g(S))) \quad (152)$$

The Nemytski map is Fréchet smooth on the strong Hoelder torus group.

This allows Léandre [85] to define stochastic plots, generalizing to this case the notion of diffeology of Chen [24] and Souriau [121] (see [43] for related works).

Let us recall what a diffeology on a topological space M is. It is constituted of a collection of maps (ϕ_O, O) from any open subset O of any R^n satisfying the following requirements:

- (i) If $j : O_1 \rightarrow O_2$ is a smooth map from O_1 into O_2 and if (ϕ_{O_2}, O_2) is a plot, $(\phi_{O_2} \circ j, O_1)$ is still a plot called the composite plot.
- (ii) The constant map is a plot.
- (iii) If U_1 and U_2 are two open disjoint subsets of R^n and if (ϕ_{O_1}, O_1) and (ϕ_{O_2}, O_2) are two plots, the union map $\phi_{O_1 \cup O_2}$ realizes a plot from $O_1 \cup O_2$ into M .

This allows Chen and Souriau to define a form. A form σ is given by data of forms $\phi_O^* \sigma$ on O associated to each plot (ϕ_O, O) . The system of forms over U $\phi_U^* \sigma$ has moreover to satisfy the following requirement: if $(\phi_{O_2} \circ j, O_1)$ is a composite plot, $(\phi_{O_2} \circ j)^* \sigma$ is equal to $j^* \phi_{O_2}^* \sigma$.

Let us recall if σ_O is a r-form on O and if $j : O' \rightarrow O$ is smooth application $j^* \sigma(X'_1, \dots, X'_r) = \sigma(DjX'_1, \dots, DjX'_r)$ for vector fields on O' .

The exterior derivative $d\sigma$ of σ is given by the data $d\phi_O^* \sigma$.

The main example of Souriau is the following: let M be a manifold endowed with the equivalence relation \sim . We can consider the quotient space \tilde{M} . Let π be the projection from M onto \tilde{M} . A map $\tilde{\phi}$ from an open subset U of a finite-dimensional linear space is a plot with values in \tilde{M} if, by definition, there is a smooth lift ϕ from U into M such that $\tilde{\phi} = \pi \circ \phi$.

Definition 20 A stochastic plot of dimension n on the strong Hoelder torus group is given by a countable family (O, ϕ_i, Ω_i) where O is an open subset of R^n such that:

- (i) The Ω_i constitute a measurable partition of $T_{\epsilon, *}^2(G)$.
- (ii) $\phi(u)(S) = f_i(u, S, g(S))$ where f_i is a smooth function over $O \times T^2 \times R^m$ with bounded derivatives of all orders into R^m .
- (iii) Over Ω_i , $\phi_i(u)(\cdot)$ belongs to the torus group.

When there is no ambiguity, we call a stochastic plot ϕ .

We identify two stochastics plots $(O, \phi_i^1, \Omega_i^1)$ and $(O, \phi_j^2, \Omega_j^2)$ if $\phi_i^1 = \phi_j^2$ almost surely over $\Omega_i^1 \cap \Omega_j^2$.

This allows, following the lines of Chen [24] and Souriau [121] to define a stochastic form [85] (almost surely defined!):

Definition 21 A stochastic r-form σ_{st} with values in R is given by the following data: To each plot ϕ with source O , we associate a random smooth r-form $\phi^* \sigma_{st}$ on O which satisfy to the following requirements:

- (i) Let j be a smooth map from O^1 into O^2 and let ϕ^2 a plot with source O^2 . We can consider the composite plot $\phi^1 = \phi^2 \circ j$ with source O^1 . Then, almost surely:

$$\phi^{1*} \sigma_{st} = j^* \phi^{2*} \sigma_{st} \quad (153)$$

- (ii) If $(O, \phi_i^1, \Omega_i^1)$ and $(O, \phi_j^2, \omega_j^2)$ are two stochastic plots such that $\phi_i^1 = \phi_j^2 \circ \psi$ on a set of probability different to 0 for a given measurable transformation of some Ω_i^1 into some Ω_j^2 , then almost surely as smooth forms on O :

$$\phi_i^{1*} \sigma_{st} = \phi_j^{2*} \sigma_{st} \circ \psi \quad (154)$$

We can define on the strong Hoelder torus group the standard de Rham cohomology groups, for forms smooth in Fréchet sense, because it is a Banach manifold. It is not heavy to do that, because there are partition of unity on the strong Hoelder torus group.

We can define the stochastic exterior derivative for stochastic forms σ_{st} . It is defined by the collection of $d\phi^*\sigma_{st}$ where we consider the collection of stochastic plots ϕ . (See [24] and [121] in the deterministic context.)

[85] proves this theorem:

Theorem 22 *The stochastic de Rham cohomology groups of the strong Hoelder torus group are equal to the deterministic Fréchet de Rham cohomology groups of the strong Hoelder torus group.*

The proof is based upon the fact there are partition of unity on the strong Hoelder torus group.

Let us recall quickly the material used in order to prove this theorem, which was used for a finite dimensional manifold in order to show that the various classical cohomologies of a finite dimensional manifold are equal [124].

Let M be a topological space. It will be later the strong Hoelder torus group.

Definition 23 *A presheaf $P = \{\Gamma(U; P); \rho(U, V)\}$ of R -vector spaces is a collection of vector spaces $\Gamma(U; P)$ indexed by the open subsets U of M and a restriction map $\rho(V; U) : \Gamma(V; P) \rightarrow \Gamma(U; P)$ for $U \subseteq V$ such that for $U \subseteq V \subseteq W$ we have:*

$$\rho(U; W) = \rho(U; V) \circ \rho(V; W) \quad (155)$$

Definition 24 *A sheaf $\tilde{S} = \{\Gamma(U; \tilde{S}); \rho(U; V)\}$ of R -vector spaces is a presheaf such that for any cover U_i by open subsets of M , the following two properties are checked:*

- (i) *If the restriction to $U \cap U_i$ of a section f belonging to $\Gamma(U; \tilde{S})$ equal the restriction to $U \cap U_i$ of another section g of $\Gamma(U; \tilde{S})$, then $f = g$.*
- (ii) *Let us give a system of section f_i of $\Gamma(U_i; \tilde{S})$ such that the restriction to $U_i \cap U_j$ of f_i is equal to the restriction to $U_i \cap U_j$ of f_j . There exists a unique $f \in \Gamma(U; \tilde{S})$ such that its restriction to $U_i \cap U$ are equal to the restriction of f_i to $U \cap U_i$.*

If we replace M by the strong Hoelder torus group, we can produce two sheaves on it:

- (i) The sheaf of stochastic form in Chen-Souriau sense.
- (ii) The sheaf of Fréchet smooth forms on it. It is possible to do that because the strong Hoelder torus group is a Fréchet manifold modelled on the space of strong Hoelder map from T^2 into the Lie algebra of G .

Definition 25 *A morphism of sheaf $d : \tilde{S}' \rightarrow \tilde{S}$ is a collection of linear mappings d_u from $\Gamma(U; \tilde{S}')$ into $\Gamma(U; \tilde{S})$ which are compatible with the operations of restrictions.*

For instance, a Fréchet smooth form on the strong Hoelder torus group defines clearly a stochastic form. We deduce a morphism of the sheaf of Fréchet smooth forms into the space of stochastic forms which is the inclusion.

The stochastic exterior derivative is a morphism of the sheaf of stochastic forms as well as the traditional exterior derivative for the sheaf of Fréchet smooth forms.

Definition 26 *A morphism of sheaves $d : \tilde{S} \rightarrow \tilde{S}' \rightarrow \tilde{S}''$ is exact if for every open subset V , there exists an open subset $U \subseteq V$ such that $\text{Im}(d_U) = \text{Ker}(d_U)$.*

This means that we have a kind of Poincaré lemma. The stochastic exterior derivative is exact as well as the traditional exterior derivative on the sheaf of Fréchet smooth forms on the strong Hoelder torus group.

A sheaf over M is said to be fine if for each locally finite cover U_i by open subsets, there exists for each i an endomorphism l_i of the sheaf S such that:

- (i) $\text{Supp } l_i \subseteq U_i$.
- (ii) $\sum l_i = 1$

l_i are called partition of unity.

Since we work on the strong Hoelder torus group, the sheaf of stochastic form and of deterministic forms are fine.

Theorem 22 arises then by abstract arguments on sheaf cohomology [124].

Let us define a 1-dimensional stochastic plot l with source $[a, b]$. We can define:

$$\int_l \sigma_{st} = \int_a^b l^* \sigma_{st} \tag{156}$$

if σ_{st} is a stochastic 1-form (see [24, 121] in the deterministic case and [85] in the stochastic case).

We can consider a sum l or a different of oriented segments l^k , with oriented boundaries. We say we are in presence of a stochastic 1-dimensional cycle if the boundaries destroy. We put:

$$\int_l \sigma_{st} = \sum \int_{l^k} \sigma_{st} \tag{157}$$

Definition 27 *We say that a stochastic 1-form is Z -valued if $d\sigma_{st} = 0$ and if for any 1-dimensional stochastic cycle l , the random variable $\int_l \sigma_{st}$ belongs to Z almost-surely.*

This definition is analogous to the deterministic one, with only difference that the stochastic form are involved, as we will see, with stochastic integrals, and therefore we cannot pick-up a trajectory of $g(\cdot)$. We will perform the following hypothesis:

Hypothesis H: the torus group is connected and $\Pi^1(G) = 0$ such that the loop group is simply connected.

Namely if $\Pi(G) = 0$, $\Pi^2(G) = 0$.

In such a case, there exists a line (a 1-dimensional stochastic plot!) joining $e(\cdot)$ to $g(\cdot)$.

Definition 28 Let σ_{st} be a Z -valued 1-form. Let k be an integer. The generalized Wess-Zumino-Novikov-Witten model of level k is given by the measure on the strong Hoelder torus group:

$$d\mu_k = \exp[2\sqrt{-1}\pi k \int_l \sigma_{st}]d\mu \tag{158}$$

This is defined independently of the connecting plot l , because σ_{st} is Z -valued. This last property implies namely if two 1-dimensional stochastic plots l_1 and l_2 join $e(\cdot)$ to $g(\cdot)$, then $\int_{l_1} \sigma_{st}$ differs of $\int_{l_2} \sigma_{st}$ by a random integer.

This allows to define consistently the Wess-Zumino term $\exp[2\sqrt{-1}\pi k \int_l \sigma_{st}]$.

We will produce a stochastic interpretation of the classical examples of conformal field theory, by using Malliavin's transfer principle. This requires to define Stratonovitch integrals, since the field $g(\cdot)$ is only Hoelder. It is done in [85] and is the object of the two next theorems:

Theorem 29 (*Integral of a 1-form*): Let ω be a 1-form on G . The stochastic integral of Stratonovitch type

$$\int_{S^1} \langle \omega(g(s, t)), d_s g(s, t) \rangle \tag{159}$$

exists almost surely and is limit in L^2 of the traditional random integrals $\int_{S^1} \langle \omega(g^N(s, t)), d_s g^N(s, t) \rangle$ where g^N is a convenient polygonal approximation of the random field $g(\cdot)$.

Let us show the mail estimate in order to show the existence of this stochastic integral (where we cannot apply Itô' theory of Stochastic integral, because there is no martingale involved with the definition of this (almost-surely defined!) stochastic integral). We remark is $s < s_1 + \Delta s_1 < s_2 < s_2 + \Delta s_2$ then

$$\langle B.(s_1 + \Delta s_1, t) - B.(s_1, t), B.(s_2 + \Delta s_2, t) - B.(s_2, t) \rangle = O(\Delta s_1 \Delta s_2) \tag{160}$$

By using Stochastic Calculus, we can show that:

$$\begin{aligned} E[\langle f_1(g_1(s_1, t)), g_1(s_1, t) - g_1(\Delta s_1 + s_1, t) \rangle \\ \langle f_2(g_1(s_2, t)), g_1(s_2, t) - g_1(s_2 + \Delta s_2, t) \rangle] = O(\Delta s_1 \Delta s_2) \end{aligned} \tag{161}$$

We compute the expectation of the square of polygonal approximation of the stochastic integral associated to (159) and we arrive to the sum expression like (161) which converges when the length of the associated subdivision tend to infinity.

Theorem 30 (*Integral of a two-form*): Let ω be a 2-form on G . The Stratonovitch stochastic integral $\int_{T^2} \langle \omega(g(S)), d_s g(S), d_t g(S) \rangle$ exists and is limit in L^2 of the traditional random integrals $\int_{T^2} g^N * \omega$ where g^N is a convenient polygonal approximation of the random field $g(\cdot)$.

The proof is based upon similar arguments than (160) and (161), but the combinatoric is much more complicated.

We consider the canonical 3-form ω on the simple simply connected Lie group G which at the level of the Lie algebra of G satisfies to:

$$\omega(X, Y, Z) = \frac{1}{8\pi^2} \langle [X, Y], Z \rangle \quad (162)$$

We have the following theorem [85]:

Theorem 31

$$\sigma_{st} = \int_{T^2} \langle \omega(g(S)), d_s g(S), d_t g(S), \cdot \rangle \quad (163)$$

defines a closed Z -valued stochastic 1-form on the strong Hoelder torus group.

In order to see that, let us consider a stochastic plot (ϕ_O, O) given globally, in order to simplify, by $u \in O \rightarrow \{S \rightarrow f(u, S, g(S))\}$ (We take in order to simplify the exposition a global stochastic plot). Let X be a vector field on O .

$$\phi^* \sigma_{st} X = \int_{T^2} \langle \omega(\phi(u)(S), d_s \phi(u)(S), d_t \phi(u)(S), \partial_X \phi(u)(S)) \rangle \quad (164)$$

The integral on a stochastic cycle in the torus group gives an integral on a random (continuous) 3-dimensional cycle on G . These cycle can be approximated by random piecewise smooth cycle in G : the integral of ω on these random smooth cycle is a random integer which converges by Theorem 30 to the integral of σ_{st} on the stochastic one dimensional cycle of the torus group of σ_{st} .

The relation between the geometry of the torus group and the geometry of the loop groups comes from the following observation: from the heat-kernel measure on the torus groups, we deduced some measures on loop groups $L^t(G)$ given by $s \rightarrow g(s, t)$. We can define as before a stochastic diffeology on each loop group $L^t(G)$ as well as stochastic forms, stochastic two dimensional cycles with values in the loop group...

Theorem 32

$$\sigma_{st}^t = \int_{S^1} \omega = \int_{S^1} \langle \omega(g(S)), d_s g(s, t), \cdot, \cdot \rangle \quad (165)$$

defines a Z -valued stochastic 2-form on each $L^t(G)$.

Let (ϕ_O, O) be the above stochastic plot and X, Y two vector fields on O . We have

$$\phi^* \sigma_{st}^t(X, Y) = \int_{S^1} \langle \omega(\phi(u)(S), d_s \phi(u)(S), \partial_X \phi(u)(S), \partial_Y \phi(u)(S)) \rangle \quad (166)$$

The integral of σ_{st}^t on a stochastic cycle in the loop group gives as above a random stochastic integral on a 3-dimensional cycle on G , which is a random integer as before.

Let U_i be balls for the uniform distance on $L^t(G)$ of small radius and of center $g_i(\cdot, t)$. We can assume that they constitute a cover of $L^t(G)$. We can suppose

that δ is small enough such that we can join $g_i(\cdot, t)$ to $g(\cdot, t) \in U_i$ by a stochastic segment, and therefore we can join the unit loop to $g(\cdot, t)$ by a stochastic segment $l_i(g(\cdot, t))$. We have supposed that the loop group is simply connected. So we can find a stochastic surface in $L^t(G)$ $B_{i,j}(g(\cdot, t))$ whose boundary is $l_i(g(\cdot, t))$ and $l_j(g(\cdot, t))$ run in the opposite sense if $g(\cdot, t) \in U_i \cap U_j$. We can define, analogously to (157)

$$\rho_{i,j}(g(\cdot, t)) = \exp[-\sqrt{-1}2\pi k \int_{B_{i,j}(g(\cdot, t))} \sigma_{st}^t] \quad (167)$$

We get, since σ_{st}^t is Z -valued:

– On $U_i \cap U_j$, almost surely:

$$\rho_{i,j}\rho_{j,i} = 1 \quad (168)$$

– On $U_i \cap U_j \cap U_k$, almost surely:

$$\rho_{i,j}\rho_{j,k}\rho_{k,i} = 1 \quad (169)$$

Namely $B_{i,j}(g(\cdot, t)) \cup B_{j,k}(g(\cdot, t)) \cup B_{k,i}(g(\cdot, t))$ has no boundary such that the integral of σ_{st}^t on it is a random integer.

This allows us to give the following definition:

Definition 33 *The Hilbert space H^t of L^2 section of the stochastic line bundle ξ^t with curvature $2\pi k\sigma_{st}^t$ on $L^t(G)$ is constructed as follows. A section α^t of ξ^t is given by a collection of random variable α_j^t on U_j such that almost surely over $U_i \cap U_j$*

$$\alpha_j^t = \alpha_i^t \rho_{i,j}(g(\cdot, t)) \quad (170)$$

Moreover, since $|\alpha_i^t| = |\alpha_j^t| = |\alpha^t|$, the Hilbert structure on H^t is given by

$$E[|\alpha^t|^2] = \|\alpha^t\|^2 \quad (171)$$

A map from the torus into G realizes a loop from S^1 into $L^1(G)$ by $t \rightarrow (s \rightarrow g(s, t))$. We can define a formal parallel τ transport along this loop for ξ^1 to ξ^1 and show that, almost surely:

$$\text{Tr } \tau = \exp[2\sqrt{-1}\pi k \int_l \sigma_{st}] \quad (172)$$

In order to see that, let us recall this basic statement [18]: let M be a simply connected manifold. Let σ be a closed 2-form, which is Z -valued. This means that the integral on any cycle of Σ is an integer. σ determines a unique line bundle with connection on M whose curvature is $2\pi\sigma$. Let l be a loop in M . Since M is simply connected, l is the boundary of a surface Σ . Then the holonomy of this line bundle for the given connection is given by:

$$\tau = \exp[2\pi\sqrt{-1} \int_{\Sigma} \sigma] \quad (173)$$

We remark, since σ is Z -valued that this expression does not depend of the chosen surface Σ whose boundary is l .

This explains, following Gawedzki, the relation between the torus group and the loop group.

Léandre [89] has done a version to this theory to the case of the $1 + n$ punctured sphere: it is still a $1 + 2$ dimensional field theory. On the sphere, there is one input loop and n output loops. Léandre adds some collars to the sphere. He get a heat-kernel measure for fields parametrized by $\Sigma(1 + n)$ which are only Hoelder. Moreover, the law of each loops at the boundary are the same and they are independents, because we have added these colors. By doing as in [85], this allows, by using stochastic integrals, to realize $\Sigma(1, n)$ as a map from $H^{\otimes n}$ into H by using the fusion property (147). Moreover, the random fields parametrized by $\Sigma(1, n)$ are realized by sewing elementary pants. Moreover, there is a natural map

$$\Sigma(1, n) \times \Sigma(1, r_1) \times \cdots \times \Sigma(1, r_n) \rightarrow \Sigma(1, \sum r_i) \quad (174)$$

by sewing the exits loops of $\Sigma(1, n)$ on the input loops of $\Sigma(1, r_i)$. Léandre [89] showed, via the Markov property on the field on the sewing loops, that this operation of sewing punctured spheres along there boundaries is compatible with the natural composition maps:

$$\begin{aligned} \text{Hom}(H^{\otimes n}, H) \times \text{Hom}(H^{\otimes r_1}, H) \times \cdots \times \text{Hom}(H^{\otimes r_n}, H) \\ \rightarrow \text{Hom}(H^{\otimes \sum r_i}, H) \end{aligned} \quad (175)$$

This statements say that the collection of $\text{Hom}(H^{\otimes n}, H)$ is an operad. Archetypes of operads are the set of trees. Relation between operads and two dimensional field theories was pioneered by Kimura-Stasheff-Voronov [68] and Huang-Lepowsky [58].

In [89], the geometry of $\Sigma(1, n)$ is fixed. Let us consider the case of a two dimensional field theory. Σ is a Riemannian surface with exit and input boundary loops endowed with the canonical metric on S^1 on each on the connected components of the boundary [93].

Let us consider $\bar{\Sigma}$ got from Σ by sewing disk along the boundaries. $\bar{\Sigma}$ has a canonical metric, inherited from Σ . Let $\Delta_{\bar{\Sigma}}$ be the Laplace Beltrami operator on $\bar{\Sigma}$. Let $H_{\bar{\Sigma}}$ the Hilbert space of maps h from $\bar{\Sigma}$ into R such that:

$$\int_{\bar{\Sigma}} (\Delta_{\bar{\Sigma}}^k + 1)h(S)(\Delta_{\bar{\Sigma}}^k + 1)h(S)dg_{\bar{\Sigma}}(S) < \infty \quad (176)$$

where $dg_{\bar{\Sigma}}(S)$ denotes the Riemannian measure on $\bar{\Sigma}$.

Let $B_{\bar{\Sigma}, t}$ be the Brownian motion with values in $H_{\bar{\Sigma}}$. If k is big enough independent from r , $(t, S) \rightarrow B_{\bar{\Sigma}, t}(S)$ is continuous in $t \in [0, 1]$ and C^r in $S \in \bar{\Sigma}$ (see [93]).

Let $\bar{S}_1 = [0, 1] \times S_1$ where we sew disk along the boundary. \bar{S}_1 inherits a canonical Riemannian structure. Let $H_{\bar{S}_1}$ be the Hilbert space of maps from \bar{S}_1 into R such that

$$\int_{\bar{S}_1} (\Delta_{\bar{S}_1}^k + I)h(S)(\Delta_{\bar{S}_1}^k + 1)h(S)dm_{\bar{S}_1}(S) < \infty \quad (177)$$

Let $B_{\overline{\Sigma}_1, t}$ be the Brownian motion with values in $H_{\overline{\Sigma}_1}$. Let $g_{\Sigma}(S)$ be a map from Σ into $[0, 1]$ equal to 1 on Σ where we have removed the output collars $[0, 1/2[\times\Sigma_2$ and where we have removed the input collars $]1/2; 1] \times \Sigma_1$. We suppose that g_{Σ} is equal to zero on a neighborhood of the boundaries of Σ .

Let g^{out} be a smooth map from $[0, 1/2]$ into $[0, 1]$ equal to 0 only in 0 and equal to 1 in a neighborhood of $1/2$. Let g^{in} be a smooth map from $[1/2, 1]$ equal to 0 only in 1 and equal to 1 in a neighborhood of $1/2$.

We consider the Gaussian random field parametrized by $\Sigma \times [0, 1]$:

$$B_{\Sigma, \cdot}(\cdot) = g_{\Sigma}(\cdot)B_{\overline{\Sigma}, \cdot}(\cdot) + \sum_{\text{in}} g^{\text{in}} B_{\overline{\Sigma}_1, \cdot}^{\text{in}}(\cdot) + \sum_{\text{out}} g^{\text{out}} B_{\overline{\Sigma}_1, \cdot}^{\text{out}}(\cdot) \quad (178)$$

where we take independent Brownian motion on $H_{\overline{\Sigma}_1}$ which are independent of the Brownian motion $B_{\overline{\Sigma}}$. We have a body process and some boundary processes which are independent themselves and of the body process.

An object $\Sigma_{\text{tot}, k} = (\Sigma_1 \cup \Sigma_2 \dots \cup \Sigma_k)$ is constructed inductively as follows: Σ_1 is a Riemann surface constructed as before. $\Sigma_{\text{tot}, k+1}$ is constructed from $\Sigma_{\text{tot}, k}$ where we sew some exit boundaries of $\Sigma_{\text{tot}, k}$ along some input boundaries of Σ_{k+1} . Let us remark that in the present theory, we don't consider $\Sigma_{\text{tot}, k}$ as a Riemannian manifold, but as the sequence $(\Sigma_1, \dots, \Sigma_k)$ and the way we sew Σ_{k+1} to $\Sigma_{\text{tot}, k}$ inductively. Namely, if we consider the random fields parametrized by $\Sigma_{\text{tot}, k} \times [0, 1]$ considered as a global Riemannian manifold done by (176), it is different from the random field $B_{\Sigma_{\text{tot}, k}}$ constructed as below. In particular, the sewing collars in $\Sigma_{\text{tot}, k}$ are independent in the construction below, and are not independent in the construction (176).

We can construct inductively $B_{\Sigma_{\text{tot}, k+1}}$ as follows: if $k = 1$, it is B_{Σ} . $B_{\Sigma_{k+1}}$ is constructed from Brownian motion independent of those which have constructed $B_{\Sigma_{\text{tot}, k}}$, except for the Brownian motions in the input boundaries of $B_{\Sigma_{k+1}}$ which coincide with the Brownian motion in the output boundaries of $\Sigma_{\text{tot}, k}$ which are sewed to the corresponding input boundaries of Σ_{k+1} . By this procedure, if $S \in \Sigma_{\text{tot}}$, we get a process $(t, S) \rightarrow B_{\Sigma_{\text{tot}, t}}(S)$ which is continuous in t and C^r in $S \in \Sigma_{\text{tot}}$.

Let G be the compact simply connected Lie group. We consider Airault-Malliavin equation [93]

$$d_t g_{\Sigma_{\text{tot}, t}}(S) = g_{\Sigma_{\text{tot}, t}}(S) \sum e_i d_t B_{\Sigma_{\text{tot}}}^i(S) \quad (179)$$

starting from e . $B_{\Sigma_{\text{tot}}}^i$ are independent copies of $B_{\Sigma_{\text{tot}}}$ and e_i an orthogonal basis of the Lie algebra of G .

Theorem 34 *If k is big enough, the random field parametrized by Σ_{tot} $S \rightarrow g_{\Sigma_{\text{tot}, 1}}(S)$ is C^r . Moreover the restriction to this random field to the connected components of the boundary of Σ_{tot} are independents and have the same law.*

Let us recall some geometrical background about the Wess-Zumino-Novikov-Witten model [42]. Let Σ be an oriented surface with boundaries. Let g be a C^r map from Σ into G conveniently extended into a map $g_t(S)$ from $[0, 1] \times \Sigma$ into

G such that $g_0(z) = e$. We define the Wess-Zumino term:

$$W_\Sigma(g) = -1/6 \int_{[0,1] \times \Sigma} \langle g^{-1}dg \wedge [g^{-1}dg \wedge g^{-1}dg] \rangle \quad (180)$$

where \langle, \rangle is the canonical normalized Killing form on the Lie algebra of G . We suppose that the 3-form which is integrated in (180) represents an element of $H^3(G; Z)$ (see [42] for this hypothesis). $\exp[2\pi\sqrt{-1}W_\Sigma(g)]$ can be identified canonically to an element of $\xi_{\partial\Sigma, \partial g}$ where ξ is an Hermitian line bundle over the set of C^r maps from $\partial\Sigma$ into G . Let $\partial\Sigma_i$ be the oriented connected components of $\partial\Sigma$. We have a canonical inclusion map π_i from $\partial\Sigma_i$ in $\partial\Sigma$. We deduce from it a map $\bar{\pi}_i$ from the set of maps from $\partial\Sigma$ in G into the set of maps from $\partial\Sigma_i$ into G . Let ξ_i be the hermitian bundle on the set of maps from Σ_i into G constructed in [42]. $\xi = \otimes \bar{\pi}_i^* \xi_i$ endowed with its natural metric inherited from each ξ_i . We denote it $\otimes_{\text{exit}} \xi \otimes_{\text{in}} \xi$. Moreover, we can realize this expression as a map from the tensor products of Hermitian line bundle ξ over the exit loop groups defined by restricting the field over each exit boundary to the tensor product of Hermitian line bundles ξ over the input loop groups defined by restricting the field over each connected component of the input boundary. Therefore $\exp[2\pi\sqrt{-1}W_\Sigma(g)]$ can be realized as an application from $\otimes_{\text{exit}} \xi$ into $\otimes_{\text{in}} \xi$ of modulus 1. This application is consistent with the operation of sewing surface.

Let H' be the Hilbert space of section of ξ over the C^r loop group $L^r(G)$ endowed with the law arising from restricting the field to one boundary loops. Let Σ_i be such a boundary loop. The laws of $g_{\Sigma_{\text{tot}}, 1}(\cdot)$ restricted to each Σ_i are the same. Let $\alpha_i(g_{\Sigma_{\text{tot}}, 1}(\cdot)|_{\Sigma_i})$ a section of ξ on the set of loops defined by Σ_i . We put

$$\|\alpha_i\|_{H'_i}^2 = E[|\alpha_i(g_{\Sigma_{\text{tot}}, 1}(\cdot)|_{\Sigma_i})|^2] \quad (181)$$

Let $L^2([0, 1] \times \Sigma_i)$ be the Hilbert space of L^2 functionals with respect of $g_{V_{\text{tot}}, \cdot}(\cdot)$ restricted to $[0, 1] \times \Sigma_i$. We put $H_i = H'_i \otimes L^2([0, 1] \times V_i)$. We get always the same Hilbert space H independent of the choice of Σ_{tot} .

Definition 35 $A(\Sigma_{\text{tot}}, g_{\Sigma_{\text{tot}}})$ is the operator from $\otimes_{\text{out}} H$ into $\otimes_{\text{in}} H$ where we put the tensor product along respectively the connected components of the exit boundary of Σ_{tot} and of the input boundaries of Σ_{tot} defined as follows: let α_i be a section of ξ at the i^{th} connected component of the exit boundary:

$$\begin{aligned} & A(\Sigma_{\text{tot}}, g_{\Sigma_{\text{tot}}}) \otimes_{\text{out}} \alpha_i \\ &= E[\exp[2\pi\sqrt{-1}W_{\Sigma_{\text{tot}}}(g_{\Sigma_{\text{tot}}, 1})] \otimes_{\text{out}} \alpha_i | B'([0, 1] \times \Sigma_1)] \end{aligned} \quad (182)$$

where $B'([0, 1] \times \Sigma_1)$ is the σ -algebra spanned by the random field $g_{\Sigma_{\text{tot}}, \cdot}(\cdot)$ restricted to the input data $[0, 1] \times \Sigma_1$.

Let $(\Sigma_{\text{tot}}^1, g_{\Sigma_{\text{tot}}^1}^1)$ and $(\Sigma_{\text{tot}}^2, g_{\Sigma_{\text{tot}}^2}^2)$ and $(\Sigma_{\text{tot}}, g_{\Sigma_{\text{tot}}})$ got by sewing Σ_{tot}^1 along some exit boundaries coinciding with some input boundaries of Σ_{tot}^2 . We call the sewing boundary $\tilde{\Sigma}$ in Σ_{tot} . By Markov property of the random field [71, 73, 105, 66, 116], we deduce [93]:

Theorem 36 *We have:*

$$A(\Sigma_{\text{tot}}, g_{\Sigma_{\text{tot}}}) = A(\Sigma_{\text{tot}}^1, g_{\Sigma_{\text{tot}}^1}) \circ A(\Sigma_{\text{tot}}^2, g_{\Sigma_{\text{tot}}^2}) \quad (183)$$

where the composition goes for the Hilbert spaces which arises from the sewing boundaries.

Let us do a brief history of Markov property for random fields.

Let (Ω, F, P) be a probability space and $x(S)$ be a Gaussian continuous centered random field with parameter space a finite dimensional manifold M equipped with a Riemannian distance d .

If O is an open subset of M , we define

$$B(O) = \sigma(x(S); S \in O) \quad (184)$$

and for a closed subset D , we define:

$$B(D) = \bigcap_{\epsilon > 0} B(D_\epsilon) \quad (185)$$

where $D_\epsilon = \{S \in M, \inf_{S' \in D} d(S, S') < \epsilon\}$

Definition 37 *A random field has the Markov property with respect to an open set O if for all $B(\overline{O})$ -measurable functional F :*

$$E[F|B(O^c)] = E[F|B(\partial O)] \quad (186)$$

A random field is G -Markov if it has the Markov property with respect to all open set O .

Markov property with respect to O is equivalent to the following statement: for any event A_1 $B(\overline{O})$ -measurable and for any event A_2 $B(O^c)$ -measurable:

$$P(A_1 \cap A_2 | B(\partial O)) = P(A_1 | B(\partial O)) P(A_2 | B(\partial O)) \quad (187)$$

Let us recall that the reproducing Hilbert space H of the continuous Gaussian random field is given as follows: if F is a linear random variable of the Gaussian random field, we put:

$$f_F(S) = E[Fx(S)] \quad (188)$$

and

$$\langle f_F, f_{F'} \rangle = E[FF'] \quad (189)$$

If $e_S(S')$ is the covariance of the continuous Gaussian random field:

$$E[x(S)x(S')] = e_S(S') \quad (190)$$

such that

$$h(S) = \langle h, e_S(\cdot) \rangle \quad (191)$$

(h is the generic element of the reproducing Hilbert space H of the random field).

Let us recall ([73] Theorem 5.1):

Theorem 38 *A random continuous Gaussian field $x(\cdot)$ with reproducing Hilbert space H is G -Markov if the two following conditions are checked:*

- (i) *For all h_1 and h_2 with support disjoint $\langle f_1, f_2 \rangle = 0$.*
- (ii) *If $H \in H$ is such that $f = f_1 + f_2$ with disjoint supports, then f_1 and f_2 belong to H .*

In particular, we have consider in this paper Gaussian G -Markov fields.

We were considering previously 1 + 2 dimensional field theory, and the associated heat kernel measure and the case where the loop space is simply connected. We will consider now 1 + 1 random fields (that is diffusion processes on loop spaces), with a non simply connected loop space. We recall briefly the construction of the Brownian pants of Brzezniak-Léandre [21] and Léandre [92].

We consider the compact Riemannian manifold M of dimension d isometrically imbedded in R^m . We introduce the Hilbert space H_1 of the set of loops in R^m such that:

$$\int_0^1 |\gamma(s)|^2 ds + \int_0^1 |d/ds \gamma(s)|^2 ds = \|\gamma\|^2 < \infty \tag{192}$$

Moreover, the couple if $s \neq s'$ $t \rightarrow (B_t(s), B_t(s'))$ realizes a non degenerated Brownian motion over $R^m \times R^m$, although $t \rightarrow B_t(s)$ and $t \rightarrow B_t(s')$ are not independent: the covariance matrix of $B_t(s)$ and $B_t(s')$ is not degenerated. This comes from the fact that the two linear maps from H_1 into R^m $\gamma(\cdot) \rightarrow \gamma(s)$ and $\gamma(\cdot) \rightarrow \gamma(s')$ are independents. The family of Stratonovitch equations

$$d_t x_t(s) = \Pi(x_t(s)) d_t B_t(s); \quad x_0(s) = x \tag{193}$$

has a meaning. We recall [92] that $(s, t) \rightarrow x_t(s)$ has almost surely a version which is $1/2 - \epsilon$ Hoelder for all ϵ . This comes from the fact that the Green kernel of this theory are computed by (149) and satisfy Hypothesis H of Section 4.

Let $s_1 < s_2$ be two times. We constrain the elliptic diffusion $t \rightarrow (x_t(s_1), x_t(s_2))$ to be equal to y at time 1.

Let us recall that if we consider an elliptic diffusion $\tilde{y}_t(\tilde{x})$ over a compact manifold \tilde{M} , it has an heat kernel $q_t(\tilde{x}, \tilde{y})$

$$E[f(\tilde{y}_t(\tilde{x}))] = \int_{\tilde{M}} q_t(\tilde{x}, \tilde{y}) f(\tilde{y}) dg(\tilde{y}) \tag{194}$$

satisfying the estimate:

$$|\text{grad} \log q_t(\tilde{x}, \tilde{y})| \leq C \frac{\tilde{d}(\tilde{x}, \tilde{y})}{t} \tag{195}$$

for \tilde{y} close to \tilde{x} for the associated Riemannian metric and the natural Riemannian distance \tilde{d} associated to the elliptic diffusion (see [17, 107]). Let us recall that if the stochastic differential equation of the elliptic diffusion is given by

$$d\tilde{y}_t(\tilde{x}) = \sum \tilde{X}_i(\tilde{y}_t(\tilde{x})) d\tilde{w}_t^i + \tilde{X}_0(\tilde{y}_t(\tilde{x})) dt \tag{196}$$

over the compact manifold, the bridge between \tilde{x} and \tilde{y} (that is the diffusion constrained in time 1 to be \tilde{y}) satisfies to the following stochastic differential equation (in Stratonovitch sense):

$$d\tilde{y}_t(\tilde{x}, \tilde{y}) = \sum \tilde{X}_i(\tilde{y}_t(\tilde{x}, \tilde{y}))(d\tilde{w}_t^i + \langle \tilde{X}_i(\tilde{y}_t(\tilde{x}, \tilde{y})), \text{grad log } q_{1-t}(\tilde{y}_t(\tilde{x}, \tilde{y}), \tilde{y}) \rangle dt) + \tilde{X}_0(\tilde{y}_t(\tilde{x}, \tilde{y}))dt \quad (197)$$

(see [17, 107]). This means that we transform $d\tilde{w}_t^i$ into $d\tilde{w}_t^i + \alpha_t^i dt$ by using the equation (196). By the estimate (195), we have:

$$E\left[\int_0^1 |\alpha_t^i| dt\right] < \infty \quad (198)$$

We write

$$\begin{aligned} B_t(s_2) &= \alpha(s_1, s_2)B_t(s_1) + \beta(s_1, s_2)B_t(s_1, s_2) \\ B_t(s) &= \alpha(s_1, s_2, s)B_t(s_1) + \beta(s_1, s_2, s)B_t(s_1, s_2) + \gamma(s_1, s_2, s)B_t(s_1, s_2, s) \end{aligned} \quad (199)$$

where the Brownian motion $B_t(s_1, s_2, s)$ is independent of the Brownian motions $B_t(s_1)$ and $B_t(s_1, s_2)$. Conditioning by $x_1(s_1) = x_1(s_2) = y$ is nothing else to do the following transformation in (199).

$$\begin{aligned} d\tilde{B}_t(s) &= \alpha(s_1, s_2, s)(dB_t(s_1) + \alpha_t^1(s_1, s_2)dt) \\ &\quad + \beta(s_1, s_2, s)(dB_t(s_1, s_2) + \alpha_t^2(s_1, s_2)dt) \\ &\quad + \gamma(s_1, s_2, s)dB_t(s_1, s_2, s) \end{aligned} \quad (200)$$

We have:

Lemma 39 *We can suppose (see (198)) that $\int_0^1 |\alpha_t^i| dt < K$ (**Hypothesis K**). Under this condition have:*

$$E[|x_t(s) - x_t(s')|^p | x_1(s_1) = x_1(s_2) = y] \leq C|s - s'|^{p/2} \quad (201)$$

By using the Kolmogorov lemma (see [103]), we deduce that there exists an Hoelder version of the random field $(t, s) \rightarrow x_t(s)$ where we have conditioned by $x_1(s_1) = x_1(s_2) = y$. The loop $s \rightarrow x_1(s)$ is splitted in two loops $s \rightarrow x_1^1(s)$ and $s \rightarrow x_1^2(s)$ starting from y and satisfying the estimates (201) if (**Hypothesis K**) is satisfied. Following the idea of Brzezniak-Léandre [21] and Léandre [92], we introduce two others Brownian motions with values in the Hilbert space H_1 , $B_t^1(\cdot)$ and $B_t^2(\cdot)$, independent of each others and independent of the first Brownian motion $B_t(\cdot)$. We consider the equations after time 1

$$d_t x_{t+1}^1(s) = \Pi(x_{t+1}^1(s))dB_t^1(s) \quad (202)$$

starting from the little loop $x_1^1(\cdot)$ and

$$d_t x_{t+1}^2(s) = \Pi(x_{t+1}^2(s))dB_t^2(s) \quad (203)$$

starting from the second little loop $x_1^2(\cdot)$. We have:

Lemma 40 *If (Hypothesis K) is satisfied, we have:*

$$E[|x_t^1(s) - x_t^1(s')|^p] \leq C|s - s'|^{p/2} \quad (204)$$

and we have

$$E[|x_t^2(s) - x_t^2(s')|^p] \leq C|s - s'|^{p/2} \quad (205)$$

Definition 41 *the random pant is constituted for $t \leq 1$ by the random field $(t, s) \rightarrow x_t(s)$ with the constrain $x_1(s_1) = x_1(s_2) = y$ and for $t > 1$ by the couple of diffusion processes $t \rightarrow (x_t^1(\cdot), x_t^2(\cdot))$.*

There are one input boundary at $t = 0$ and two output boundaries to the pant $(x_2^1(\cdot), x_2^2(\cdot))$.

Let us consider the product of loop spaces $L(M) \times L(M)$. We endow it with the probability law of $(x_2^1(\cdot), x_2^2(\cdot))$. We will construct a line bundle over $L(M) \times L(M)$, by using the arguments of Gawedzki [45]. We won't suppose that the loop space is simply connected, because our construction is motivated by Deligne cohomology [18]. Namely, in the case where the loop space is simply connected, the following construction are not useful because in such a case a (complex) line bundle is defined by its curvature, as we have seen before. The following constructions are interesting only when the loop space is not simply connected.

Let O_α be a cover of M by convex contractibles open subsets of M , such that $O_{\alpha_1, \alpha_2} = O_{\alpha_1} \cap O_{\alpha_2}$, $O_{\alpha_1, \alpha_2, \alpha_3} = O_{\alpha_1} \cap O_{\alpha_2} \cap O_{\alpha_3}$ and $O_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} = O_{\alpha_1} \cap O_{\alpha_2} \cap O_{\alpha_3} \cap O_{\alpha_4}$.

Let $g_{\alpha_1, \alpha_2, \alpha_3}$ be a family of smooth functions S^1 -valued which are multiplicatively antisymmetric in $\alpha_1, \alpha_2, \alpha_3$ and such that

$$g_{\alpha_1, \alpha_2, \alpha_3} g_{\alpha_0, \alpha_2, \alpha_3}^{-1} g_{\alpha_0, \alpha_1, \alpha_3} g_{\alpha_0, \alpha_1, \alpha_2}^{-1} = 1 \quad (206)$$

over $O_{\alpha_0, \alpha_1, \alpha_2, \alpha_3}$.

Also, let $\eta_{\alpha_1, \alpha_2} = -\eta_{\alpha_2, \alpha_1}$ be a smooth real 1-form over O_{α_1, α_2} such that:

$$\eta_{\alpha_1, \alpha_2} - \eta_{\alpha_0, \alpha_2} + \eta_{\alpha_0, \alpha_1} = 1/ig_{\alpha_0, \alpha_1, \alpha_2}^{-1} dg_{\alpha_0, \alpha_1, \alpha_2} \quad (207)$$

on $O_{\alpha_0, \alpha_1, \alpha_2}$. Finally, we suppose that ω_α is a real 2-form defined on O_α such that:

$$\omega_{\alpha_1} - \omega_{\alpha_0} = d\eta_{\alpha_0, \alpha_1} \quad (208)$$

on O_{α_0, α_1} . These data define an element of the second Deligne hypercohomology group of the manifold (see [18], p. 250–251). If we look at the 3-form $d\omega_\alpha = \omega$, they patch together by (208) in order to give a closed 3-form ω on M .

Consider a system (l, v) which constitutes a triangulation of the circle S^1 such that b is an edge and $v \in \partial b$ is one of its vertex. To each edge, we associate an element α_b and to each vertex v we associate an number α_v such that the following hold: we consider the set of loops γ such that for each edge b $\gamma(b) \subseteq O_{\alpha_b}$ and such for all vertices $\gamma(v) \in O_{\alpha_v}$. This defines an open subset

$$U_{A, \alpha} = \{\gamma : S^1 \rightarrow M | \gamma(b) \subseteq O_{\alpha_b}, \gamma(v) \in O_{\alpha_v} \text{ for each } (b, v) \in A\} \quad (209)$$

If we consider the product of the loop space, we consider the product $U_{A,\alpha} \times U_{A',\alpha'}$ which constitutes a cover by open subsets of the product of the loop space.

We would like to define a system of transition maps of $(U_{A_1\alpha_1} \times U_{A'_1\alpha'_1}) \cap (U_{A_2\alpha_2} \times U_{A'_2\alpha'_2})$. Let us define the refined triangulation of both triangulation A_1 and A_2 by (\bar{b}, \bar{v}) , $\bar{v} \in \partial\bar{b}$, the triangulation A_1 by (b_i, v_i) , $v_i \in \partial b_i$ and the second triangulation by (b_2, v_2) , $v_2 \in \partial b_2$. Let us put $\alpha_b^1 = \alpha_{b_1}^1$ and $\alpha_b^2 = \alpha_{b_2}^2$. If \bar{v} is a vertex of the new triangulation, we put $\alpha_{\bar{v}}^1 = \alpha_{v_1}^1$ if $\bar{v} = v_1$ and $\alpha_{\bar{v}}^1 = \alpha_{b_1}^1$ if \bar{v} is in the interior point of the interval b_1 . We define $\alpha_{\bar{v}}^2$ analogously. The system of transition functionals of the stochastic line bundle over $L(M) \times L(M)$ is defined by

$$\rho = \rho_{A_1, \alpha_1, A_2, \alpha_2}(x_2^1) \rho_{A'_1, \alpha'_1, A'_2, \alpha'_2}(x_2^2) \quad (210)$$

where

$$\rho_{A_1, \alpha_1, A_2, \alpha_2}(x_2^1) = \exp\left[i \sum_b \int_b \eta_{\alpha_b^1} \alpha_b^2 (d_s x_2^1(s))\right] \prod_{\bar{v}, \bar{b}, \bar{v} \in \partial\bar{b}} \frac{g_{\alpha_{\bar{v}}^1 \alpha_{\bar{v}}^2} \alpha_{\bar{b}}^2}{g_{\alpha_{\bar{v}}^1 \alpha_{\bar{b}}^1} \alpha_{\bar{b}}^2}(x_2^1(v)) \quad (211)$$

and the analogous formula holds for $\rho_{A'_1, \alpha'_1, A'_2, \alpha'_2}(x_2^2)$. The transition functions are almost surely defined, due to the presence of stochastic integral in the definition of them. So we cannot define $\xi = \xi_1 \otimes \xi_2$, but we will follow the lines of Definition 32 in order to define the Hilbert space of L^2 sections of it.

Definition 42 *A L^2 section of the line bundle $\xi_1 \otimes \xi_2$ over $L(M) \times L(M)$ is a system of functionals over $U_{A,\alpha} \times U_{A',\alpha'}$ $\alpha_{A,\alpha,A,\alpha'}$ submitted to the relations: almost surely, over $(U_{A_1,\alpha_1} \times U_{A'_1,\alpha'_1}) \cap (U_{A_2,\alpha_2} \times U_{A'_2,\alpha'_2})$, we get $\alpha_{A_1,\alpha_1,A'_1,\alpha'_1} = \rho_{A_2,\alpha_2,A'_2,\alpha'_2}$.*

We can define since ρ defined by (211) is of modulus 1 the norm of a section $|\alpha|$. We suppose $E[|\alpha|^2] < \infty$ in order to define the space of L^2 sections of $\xi_1 \otimes \xi_2$.

In order this definition has some consistency, we recall that almost surely over $(U_{A_1,\alpha_1} \times U_{A'_1,\alpha'_1}) \cap (U_{A_2,\alpha_2} \times U_{A'_2,\alpha'_2})$, we get:

$$\begin{aligned} \rho_{A_1, \alpha_1, A_2, \alpha_2}(x_2^1) \rho_{A_2, \alpha_2, A_1, \alpha_1}(x_2^1) &= 1 \\ \rho_{A'_1, \alpha'_1, A'_2, \alpha'_2}(x_2^2) \rho_{A'_2, \alpha'_2, A'_1, \alpha'_1}(x_2^2) &= 1 \end{aligned} \quad (212)$$

and that on $U_{A_1,\alpha_1} \cap U_{A_2,\alpha_2} \cap U_{A_3,\alpha_3}$, we get almost surely:

$$\rho_{A_1, \alpha_1, A_2, \alpha_2}(x_2^1) \rho_{A_2, \alpha_2, A_3, \alpha_3}(x_2^1) \rho_{A_3, \alpha_3, A_1, \alpha_1}(x_2^1) = 1 \quad (213)$$

This identity still works for the product of transition functions defined by $\rho_{A_1, \alpha_1, A_2, \alpha_2}(x_2^1) \rho_{A'_1, \alpha'_1, A'_2, \alpha'_2}(x_2^1)$.

This comes from the fact these relations are surely true for the deterministic loop space and that we can approach in the stochastic case the stochastic integrals which appear in the (almost surely defined!) transition function by their polygonal approximation.

In the previous definition, we have supposed that the section is almost surely defined over the product of random loops (x_2^1, x_2^2) and is (x_2^1, x_2^2) measurable.

We can suppose that $\alpha_{A,\alpha,A',\alpha'}$ depends from all the random pants, or if we choose $1 \leq t \leq 2$, $1 \leq t \leq 2$, it depends from all the paths between t and 2. $\alpha_{A,\alpha,A',\alpha'}$ becomes an element of $L^2(\text{pant}) \otimes L^2(U_{A,\alpha}(x_2^1) \otimes U_{A',\alpha'}(x_2^1))$ and still satisfies to the consistency relations of Definition 41. We can define the L^2 norm of the section α . This increases the degree of freedom and is done in order to define what is the parallel transport over the random path from (x_2^1, x_2^2) into the path (x_1^1, x_1^2) . We will get a section of the bundle over $L_x(M) \times L_x(M)$, $\xi_1 \otimes \xi_2$ for the measure defined by (x_1^1, x_1^2) , but with an extra degree of freedom, that is the path between (x_2^1, x_2^2) to (x_1^1, x_1^2) . In order to define the stochastic parallel transport from a random section over $\xi_1 \otimes \xi_2$ over (x_2^1, x_2^2) to (x_1^1, x_1^2) along the path $t \rightarrow (x_t^1, x_t^2)$, we will use the double integral of the previous part.

Let us divide the time interval $[1, 2]$ into the stochastic intervals $[\tau_i, \tau_{i+1}[$ where (x_t^1, x_t^2) over $[\tau_i, \tau_{i+1}[$ the process (x_t^1, x_t^2) lives over some open subset $U_{A,\alpha} \times U_{A',\alpha'}$. We have described the time interval into a finite numbers of random intervals. Moreover, the times τ_i are stopping times.

Let us suppose that the parallel transport from $(x_{\tau_i}^1, x_{\tau_i}^2)$ to (x_1^1, x_1^2) is well defined. Let us call it $\tau_{1,\tau_i} = \tau_{1,\tau_i}^1 \otimes \tau_{1,\tau_i}^2$ (The product formula will be explained by the next considerations). If $t \in [\tau_i, \tau_{i+1}[$, we have, by using analogous theorem in the present situation than Theorem 29 and Theorem 30,

$$\begin{aligned}
\tau_{1,t} &= \tau_{1,\tau_i} \left\{ \exp[\sqrt{-1} \int_{\tau_i}^{t \wedge \tau_{i+1}} \sum_b \omega_{\alpha_b}(d_s x_t^1(s), d_t x_t^1(s)) \right. \\
&\quad \left. + \sqrt{-1} \sum_{v,b,v \in \partial b} \int_{\tau_i}^{t \wedge \tau_{i+1}} \eta_{\alpha_v, \alpha_b}(d_t x_t^1(v))] \right\} \\
&\quad \otimes \left\{ \exp[\sqrt{-1} \int_{\tau_i}^{t \wedge \tau_{i+1}} \sum_{b'} \omega_{\alpha_{b'}}(d_s x_t^2(s), d_t x_t^2(s)) \right. \\
&\quad \left. + \sqrt{-1} \int_{\tau_i}^{t \wedge \tau_{i+1}} \sum_{v',b',v' \in \partial b'} \eta_{\alpha_v, \alpha_{b'}}(d_t x_t^2(v))] \right\} \quad (214)
\end{aligned}$$

Let us explain this formula: for the smooth loop space, [45] gives the formula in term of double integral of the amplitude of this line bundle (that is a generalized parallel transport). We can deduce in the system of local charts given in (209) the Connection 1-form of this line bundle on the loop space. If Γ is a connection 1-form of a line bundle, the parallel transport is given by (57), which can be integrated since we consider a line bundle. The extension in the stochastic case in our situation gives (214).

Let us remark that by induction the parallel transport is of modulus one (214). The rules given in the previous parts of approximation of Stratonovitch integrals allow to state this theorem:

Theorem 43 *If α is a section of $L(M) \times L(M)$ for the measure of (x_2^1, x_2^2) and measurable for (x_2^1, x_2^2) , $\tau_{1,2}\alpha$ is a section of $\xi_1 \otimes \xi_2$ for the measure of (x_1^1, x_1^2) (but in an extended sense, because there are many paths joining (x_2^1, x_2^2) to (x_1^1, x_1^2)). Moreover,*

$$E[|\tau_{1,2}\alpha|^2] = E[|\alpha|^2] \quad (215)$$

We refer to [92] for this result.

Let us work in time 1. We consider the product of loop space $L_y(M) \times L_y(M)$ for the measure (x_1^1, x_1^2) and the loop $L(M)$ induced by concatenation of the two loops for the measure induced by x_1 . This induces a map π

$$L_y(M) \times L_y(M) \rightarrow L(M) \tag{216}$$

which preserves the measure. Over $L_y(M) \times L_y(M)$, we have the stochastic line bundle $\xi_1 \otimes \xi_2$ and over $L(M)$ we have the stochastic line bundle ξ defined by the previous considerations for the random loop x_1 .

For x_1 , we define a triangulation by choosing vertices s_1 and s_2 . We have $\gamma(s_1) = \gamma(s_2) = y \in O_{\alpha_y}$. We choose another triangulation (b^1, v^1) where we have chosen s_1 and s_2 among the vertices. For the first triangulation, we suppose $\gamma(b) \subseteq O_{\alpha_b}$ and $\gamma(v) \in O_{\alpha_v}$ where $O_{\alpha_{s_1}} = O_{\alpha_{s_2}} = \bar{O}$ is fixed, and for the second triangulation, we choose $\gamma(b') \subseteq O_{\alpha_{b'}}$ and $\gamma(v') \in O_{\alpha_{v'}}$ where $O_{\alpha_{s_1}} = O_{\alpha_{s_2}} = \bar{O}$ for the same open subset \bar{O} than the first triangulation.

We deduce from the previous triangulation two triangulations of $L_y(M)$ and from the second triangulation two triangulations of $L_y(M)$. The transition map for the big loop space $L(M)$ is given by (211) where we replace $d_s x_2^1(s)$ by $d_s x_1(s)$ and $x_2^1(v)$ by $x_1(v)$ for the refined triangulation of the two big triangulations of the big circle. But it is almost surely equal to the product of the two transition functions where we consider the couple of loops $(x_1^1(s), x_2^1(s))$. This shows us that the fusion property (147) $\xi_1 \otimes \xi_2 = \pi^* \xi$ is satisfied. This means that a L^2 section of ξ over the random loops x_1 for $L(M)$ corresponds naturally to a L^2 section of $\xi_1 \otimes \xi_2$ over $L_y(M) \times L_y(M)$ for the law (x_1^1, x_1^2) and the L^2 norms are conserved. We assimilate $\tau_{1,2}\alpha$ to a section over x_1 . Afterwards, we use the stochastic parallel transport from x_1 to x_0 $\tau_{0,1}$. We put $\tilde{\alpha} = \tau_{0,1}\tau_{1,2}\alpha$. Since x_0 is the constant loop $s \rightarrow x$, $\tilde{\alpha}$ is a random variable. We have [92]:

Theorem 44 $E[|\tilde{f}|^2] = E[|f|^2]$.

This comes from the fact that $\tau_{0,1}$ is a random isometry from the stochastic fiber of the bundle over the random loop x_1 to the fiber over the constant loop.

Brzezniak-Léandre [21] have used the theory of Brzezniak-Elworthy [19] in order to realize stochastic pants over a suitable Besov-Slobodetsky space $W^{\theta,p}$ of loops $\gamma(\cdot)$ in the manifold M . The pants starts from the loop $\gamma(\cdot)$ and has two end loops $x_2^1(\gamma)$ and $x_2^2(\gamma)$. Let E be the Banach space of continuous bounded functionals on $W^{\theta,p}$ and $E \otimes E$ be the Banach space of bounded continuous functionals on $W^{\theta,p} \times W^{\theta,p}$. Let

$$T(F) : \gamma(\cdot) \rightarrow E[F(x_2^1(\gamma), x_2^2(\gamma))] \tag{217}$$

if $F \in E \otimes E$. T realizes a continuous linear application from $E \otimes E$ into E . This means that the Brownian pant is Fellerian.

Brzezniak-Léandre [20] have applied the theory of Brzezniak-Elworthy [19] to some Besov-Slobodetsky space of differentiable loops, where the line bundle of Gawedzki [45] is surely defined, and have defined the stochastic parallel transport on it for the Brownian motion on differentiable loops.

Gawedzki-Reis [49] have a simpler way, with more tractable combinatorial formulas, to construct a line bundle on differentiable paths, by using bundle gerbes theory. Léandre [90] constructs the Brownian motion along the Hoelder loop space, defines a stochastic line bundle in the manner of Gawedzki-Reis and study the stochastic parallel transport of it along the path of the Brownian motion on the Hoelder loop space.

With this branching type mechanism (classical in theoretical physic [102], Léandre [94] has defined a kind of Branching process on the loop space.

In (201), we have conditioned by $x_t(s) = x_t(s')$ for two fixed time $s = s'$. When there is no cut-locus, we can conditionate by this procedure by $x_1(s) = y$ for all s , which produced a kind of Brownian bridge in infinite dimension ([L19]).

[95] produces a kind of conditioning by an infinite dimensional constrain, where we cannot apply the classical tool of Airault-Malliavin-Sugita construction.

By using tools of Airault-Malliavin-Sugita construction, Léandre [96] has produced a stochastic regularization of the Poisson-Sigma model of Cattaneo-Felder which gives an infinite dimensional analog of Klauder's regularization of Hamiltonian path integral in quantum mechanic [67]. So there are two regularization in field theory:

- (i) The first one is stochastic quantization of Parisi-Wu, which uses infinite dimensional Langevin equation.
- (ii) The second one is stochastic quantization of Klauder, which uses infinite dimensional Brownian motion of Airault-Malliavin.

Let us remark that Brylinski [18] constructed a line bundle by using category theory and gerbes theory of Grothendieck on the smooth loop space. Léandre [91] gives a stochastic interpretation of [18] by studying a stochastic line bundle on the Brownian bridge of a manifold.

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